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ON THE SCALAR FUNCTIONS OF HYPER COMPLEX NUMBERS.

SECOND PAPER.

By Henry Taber.

§ 1.

In this paper I shall denote by γ_{iik}, for $i, j, k = 1, 2, \ldots m$, the constants of multiplication of a given non-nilpotent hyper complex number system $(e_1, e_2, \ldots e_m)$.[1] We then have

$$(1) \qquad e_i e_j = \sum_{k=1}^{m} \gamma_{ijk} e_k \qquad (i, j = 1, 2, \ldots m).$$

In These Proceedings, vol. 41 (1905), p. 59, I have shown that there are two functions of the coefficients of any number

$$(2) \qquad A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

of the system $(e_1, e_2, \ldots e_m)$ constituting generalizations of the scalar function of quaternions, to which they reduce, becoming identical when $m = 4$, and, at the same time, the system (e_1, e_2, e_3, e_4) is equivalent to the system constituted by the four units of quaternions. These functions, in designation the *first* and *second scalar* of A, are defined as follows:

$$(3) \qquad S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{ijj},$$

$$(4) \qquad S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{jij},$$

and conform to *theorem I* given below. In this paper I shall employ these functions to establish a simple criterion for the existence of an invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$, and a method of determining the maximum invariant nilpotent sub system, if any exist.[2] These results are embodied in *theorem II.*

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[1] A number $A = \sum_{i=1}^{m} a_i e_i$ of any hyper complex system $(e_1, e_2, \ldots e_m)$ is *idempotent* if $A^2 = A \neq 0$; A is *nilpotent*, if $A \neq 0$ but $A^p = 0$ for some positive integer $p > 1$. A system is *nilpotent*, if it contains no idempotent number; otherwise, *non-nilpotent*. Every number of a nilpotent system is nilpotent. See B. Peirce, Am. Journ. Maths., **4**, 113, (1881); cf. H. E. Hawkes, Trans. Am. Math. Soc., **3**, 321 (1902).

Theorem I. *Let γ_{ijk}, for $i, j, k = 1, 2, \ldots m$, be the constants of multiplication of any given hyper complex number system $(e_1, e_2, \ldots e_m)$. Let*

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

be any number of the system; and let

$$S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{ijj},$$

$$S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{jij}.$$

Then both $S_1 A$ and $S_2 A$ are invariant to any linear transformation of the system: that is, if

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \qquad (i = 1, 2, \ldots m),$$

the determinant of transformation not being zero, and if

$$e'_i e'_j = \sum_{k=1}^{m} \gamma'_{ijk} e'_k \qquad (i, j = 1, 2, \ldots m),$$

and

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m = a'_1 e'_1 + a'_2 e'_2 + \ldots + a'_m e'_m,$$

then

$$S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a'_i \gamma'_{ijj},$$

$$S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a'_i \gamma'_{jij}.$$

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[2] A sub system $B_1, B_2, \ldots B_p$ of any hyper complex number system $(e_1, e_2, \ldots e_m)$ is said to be *invariant* if the product in either order of each number of $(e_1, e_2, \ldots e_m)$ and each number of $(B_1, B_2, \ldots B_p)$ belongs to the sub system, for which the necessary and sufficient conditions are

$$e_i B_j = g'_{1ij} B_1 + g'_{2ij} B_2 + \ldots + g'_{pij} B_p,$$
$$B_j e_i = g''_{1ij} B_1 + g''_{2ij} B_2 + \ldots + g''_{pij} B_p$$
$$(i = 1, 2, \ldots m;\ j = 1, 2, \ldots p).$$

An invariant sub system $(B_1, B_2, \ldots B_p)$ is an *invariant nilpotent* sub system if its units by themselves constitute a nilpotent system; and in that case is a *maximum invariant nilpotent* sub system if it contains every invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$.

If ρ is any scalar, and

$$B = b_1 e_1 + b_2 e_2 + \ldots + b_m e_m$$

any second number of the system, we have

$$S_1 \rho A = \rho S_1 A, \qquad S_2 \rho A = \rho S_2 A,$$
$$S_1 (A \pm B) = S_1 A \pm S_1 B, \qquad S_2 (A \pm B) = S_2 A \pm S_2 B,$$
$$S_1 AB = S_1 BA, \qquad S_2 AB = S_2 BA.$$

If ϵ is a modulus of the system,

$$S_1 \epsilon = 1 = S_2 \epsilon.$$

If A is nilpotent,

$$S_1 A^p = 0, \qquad S_2 A^p = 0,$$

for every positive integer p; and conversely, if either

$$S_1 A^p = 0 \qquad (p = 1, 2, \ldots m)$$

or

$$S_2 A^p = 0 \qquad (p = 1, 2, \ldots m),$$

A is nilpotent. Moreover, A is nilpotent if

$$S_1 A e_1 = S_1 A e_2 = \ldots = S_1 A e_m = 0,$$

or

$$S_2 A e_1 = S_2 A e_2 = \ldots = S_2 A e_m = 0.$$

If A is idempotent, there are $m\, S_1 A > 0$ linearly independent numbers of the system satisfying the equation

$$AX = X,$$

in terms of which every number of the system satisfying this equation can be expressed linearly, also $m S_2 A > 0$ linearly independent numbers satisfying the equation

$$XA = X,$$

in terms of which every solution of this equation can be expressed linearly.[3]

Let

$$X = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m, \tag{5}$$

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[3] See paper by the author cited above, pp. 61, 69, and 70, also Trans. Am. Math. Soc., **5**, 522, (1904).

and let the number system $(e_1, e_2 \ldots e_m)$ contain at least one number satisfying the system of equations

$$(6) \qquad S_1 X e_i = x_1 S_1 e_1 e_i + x_2 S_1 e_2 e_i + \ldots + x_m S_1 e_m e_i = 0 \qquad (i = 1, 2, \ldots m).$$

The resultant of this system being the determinant

$$(7) \qquad \Delta_1 \equiv \begin{vmatrix} S_1 e_1 e_1, & S_1 e_2 e_1, & \ldots & S_1 e_m e_1 \\ S_1 e_1 e_2, & S_1 e_2 e_2, & \ldots & S_1 e_m e_2 \\ \ldots & \ldots & \ldots & \ldots \\ S_1 e_1 e_m, & S_1 e_2 e_m, & \ldots & S_1 e_m e_m \end{vmatrix},$$

we then have $\Delta_1 = o$. Let $X = B$ be any solution of equations (6). Then, by *theorem I*, B is nilpotent. Moreover, for any number A of $(e_1, e_2, \ldots e_m)$, both BA and AB are also solutions of equations (6). For, for any number

$$Y = y_1 e_1 + y_2 e_2 + \ldots + y_m e_m$$

of $(e_1, e_2, \ldots e_m)$, we now have

$$S_1 B Y = y_1 S_1 B e_1 + y_2 S_1 B e_2 + \ldots + y_m S_1 B e_m = 0;$$

in particular,

$$S_1 (BA \cdot e_i) = S_1 (B \cdot A e_i) = 0,$$

$$S_1 (AB \cdot e_i) = S_1 (A \cdot B e_i) = S_1 (B e_i \cdot A) = S_1 (B \cdot e_i A) = 0 \qquad (i = 1, 2, \ldots m).$$

Since both BA and AB are solutions of equations (6), they are both nilpotent.

Further, since, for $1 \leqq i \leqq m$, $B e_i$ is nilpotent, it follows from *theorem I* that $S_2 B e_i = 0$, and thus any solution B of the system of equations (6) is also a solution of the system of equations

$$(8) \qquad S_2 X e_i = x_1 S_2 e_1 e_i + x_2 S_2 e_2 e_i + \ldots + x_m S_2 e_m e_i = 0 \qquad (i = 1, 2, \ldots m),$$

of which the resultant is

$$(9) \qquad \Delta_2 \equiv \begin{vmatrix} S_2 e_1 e_1, & S_2 e_2 e_1, & \ldots & S_2 e_m e_1 \\ S_2 e_1 e_2, & S_2 e_2 e_2, & \ldots & S_2 e_m e_2 \\ \ldots & \ldots & \ldots & \ldots \\ S_2 e_1 e_m, & S_2 e_2 e_m, & \ldots & S_2 e_m e_m \end{vmatrix}$$

By *theorem I* every solution of equations (8) is nilpotent. Let B' be any solution of this system of equations. Precisely as above, we may show that B' is nilpotent, and that both $B'A$ and AB' are also solutions of these equations for any number A of the system $(e_1, e_2, \ldots e_m)$; and, therefore, both $B'A$ and AB' are nilpotent. Since, in particular, for $1 \leqq i \leqq m$, $B'e_i$ is nilpotent, it follows from *theorem I* that B' is a solution of the system of equations (6).

Let now the nullity [4] of the determinant Δ_1 be m', where $0 < m' < m$. There is then a set of just m' linearly independent numbers, $B_1, B_2 \ldots B_{m'}$ of the system $(e_1, e_2 \ldots e_m)$ satisfying equations (6); therefore, just m' linearly independent numbers satisfying equations (8), whence it follows that the nullity of Δ_2 is m'. For $1 \leqq j \leqq m'$, the product of B_j in either order with any number A of the system is a solution of equations (6) and, therefore, both B_jA and AB_j are expressible linearly in terms of $B_1, B_2 \ldots B_{m'}$; otherwise, there is a set of more than m' linearly independent solutions of equations (6) which is contrary to supposition. Moreover, since

$$\begin{aligned} &S_1(\rho_1 B_1 + \rho_2 B_2 + \ldots + \rho_{m'} B_{m'})e_i \\ &\quad = \rho_1 S_1 B_1 e_i + \rho_2 S_1 B_2 e_i + \ldots + \rho_{m'} S_1 B_{m'} e_i = 0 \\ &\qquad\qquad (i = 1, 2, \ldots m), \end{aligned}$$

every number linear in the B's is a solution of equations (6), and is, therefore, nilpotent. Whence it follows that $B_1, B_2 \ldots B_{m'}$ constitute an invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$.

Further, the sub system $(B_1, B_2 \ldots B_{m'})$ contains every invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$, and is therefore the maximum invariant nilpotent sub system of the latter. For, let $(C_1, C_2 \ldots C_p)$ be any invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$. Since every number of this sub system is nilpotent, in particular,

$$S_1 C_j = 0 \qquad (j = 1, 2, \ldots p).$$

Moreover, since

$$\begin{aligned} C_j e_i &= g_{ji1} C_1 + g_{ji2} C_2 + \ldots + g_{jip} C_p \\ &\qquad (i = 1, 2, \ldots m;\ j = 1, 2, \ldots p), \end{aligned}$$

we have

$$\begin{aligned} S_1 C_j e_i &= g_{ji1} S_1 C_1 + g_{ji2} S_1 C_2 + \ldots + g_{jip} S_1 C_p = 0 \\ &\qquad (i = 1, 2, \ldots m;\ j = 1, 2, \ldots p); \end{aligned}$$

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[4] The nullity of a matrix or determinant of order m is m' if every $(m' - 1)$th minor (minor of order $m - m' + 1$) is zero but not every m'th minor (minor of order $m - m^1$). Nullity of order m' is equivalent to rank (*Rang*) $m - m'$.

and thus each of the C's is a solution of equations (6). Therefore, each of the C's is inexpressible linearly in terms of $B_1, B_2 \dots B_{m'}$.

Let

$$(10) \qquad B_j = b_{j1}e_1 + b_{j2}e_2 + \dots + b_{jm}e_m \qquad (j = 1, 2, \dots m').$$

We may take the b's to be rational functions with respect to the domain $R\,(1)$ of the constituents of Δ_1 (or of Δ_2) which are integral quadratic functions, rational with respect to R (1), of the constants of multiplication of the number system $(e_1, e_2 \dots e_m)$. If this number system *belongs to* the domain R', that is, if its constants of multiplication lie in the domain R', the b's may be so chosen as to lie in this domain. We may take the B's as m' new units of the number system. Thus let

$$(11) \qquad e'_{m-m'+j} = B_j \qquad (j = 1, 2, \dots m'),$$

and let $e'_1, e'_2 \dots e'_{m-m'}$ be any $m-m'$ numbers of $(e_1, e_2 \dots e_m)$ which constitute with the B's a set of m linearly independent numbers. By what has just been said the coefficients of the transformation

$$(12) \qquad e'_i = \tau_{i1}e_1 + \tau_{i2}e_2 + \dots + \tau_{im}e_m \quad (i = 1, 2, \dots m)$$

of the number system can be taken rational in any domain to which the number system belongs.

If the number system is transformed by the preceding substitution (12), and if we put

$$(13) \qquad \Delta'_1 = \begin{vmatrix} S_1 e'_i e'_j \\ (i, j = 1, 2, \dots m) \end{vmatrix},$$

then, since

$$S_1 e'_i e'_j = \sum_{h=1}^{m} \sum_{k=1}^{m} \tau_{ih}\tau_{jk} S_1 e_h e_k \qquad (i, j = 1, 2, \dots m)$$

we have

$$(14) \qquad \Delta'_1 = T^2 \Delta_1,$$

where T is the determinant of the substitution. Similarly, if

$$(15) \qquad \Delta'_2 = \begin{vmatrix} S_2 e'_i e'_j \\ (i, j = 1, 2, \dots m) \end{vmatrix},$$

we have

$$(16) \qquad \Delta'_2 = T^2 \Delta_2.$$

Therefore, the equations $\Delta_1 = 0$, $\Delta_2 = 0$ are invariant to any transformation of the units of the system.

Let now $\Delta_1 \neq 0$, in which case $\Delta_2 \neq 0$, and there is no number of the system satisfying equations (6), or equations (8); and, therefore, the system contains no invariant nilpotent sub system. In this case, therefore, if

$$S_1 A e_i = S_1 B e_i \qquad (i = 1, 2, \dots m),$$

we have $A = B$; otherwise, $A - B \neq 0$ is a solution of equations (6). Similarly, if

$$S_2 A e_i = S_2 B e_i \qquad (i = 1, 2, \dots m),$$

then $A = B$.

We have now the following theorem.

Theorem II. *Let* $(e_1, e_2 \dots e_m)$ *be any non-nilpotent hyper complex number system; let*

$$X = x_1 e_1 + x_2 e_2 + \dots + x_m e_m,$$

and let

$$\Delta_1 \equiv \begin{vmatrix} S_1 e_i e_j \\ (i, j = 1, 2, \dots m) \end{vmatrix}, \quad \Delta_2 \equiv \begin{vmatrix} S_2 e_i e_j \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

be the resultants, respectively, of the two systems of equations

$$(\alpha) \qquad S_1 X e_i = x_1 S_1 e_1 e_i + x_2 S_1 e_2 e_i + \dots + x_m S_1 e_m e_i = 0 \qquad (i = 1, 2, \dots m),$$

and

$$(\beta) \qquad S_2 X e_i = x_1 S_2 e_1 e_i + x_2 S_2 e_2 e_i + \dots + x_m S_2 e_m e_i = 0 \qquad (i = 1, 2, \dots m).$$

Then, if the number system is transformed by the substitution

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \dots + \tau_{im} e_m \qquad (i = 1, 2, \dots m),$$

and if

$$\Delta'_1 = \begin{vmatrix} S_1 e'_i e'_j \\ (i, j = 1, 2, \dots m) \end{vmatrix}, \quad \Delta'_2 = \begin{vmatrix} S_2 e'_i e'_j \\ (i, j = 1, 2, \dots m) \end{vmatrix},$$

we have

$$\Delta'_2 = T^2 \Delta_1, \qquad \Delta'_2 = T^2 \Delta_2,$$

where T *is the determinant of the substitution. Further, the condition, necessary and sufficient, that the number system shall contain no invariant nilpotent sub system is that* $\Delta_1 \neq 0$, *or* $\Delta_2 \neq 0$. *In this case, if either*

$$S_1 A e_i = S_1 B e_i \qquad (i = 1, 2, \ldots m)$$

or

$$S_2 A e_i = S_2 B e_i \qquad (i = 1, 2, \ldots m),$$

we have $A = B$. *If* $\Delta_1 = 0$, *then* $\Delta_2 = 0$, *and conversely; moreover, the nullity of* Δ_1 *is equal to the nullity of* Δ_2. *Every number of the system satisfying equations* (α) *is a solution of equations* (β), *and conversely. If* B *is any solution of equations* (α) (*or of equations* (β)), *then, for any number* A *of the system* $(e_1, e_2 \ldots e_m)$, *both* $B A$ *and* $A B$ *are solutions of these equations. If the nullity of* Δ_1 *is* m', *there is a set of just* m' *linearly independent solutions of equations* (α) (*or equations* (β)); *and any such set of* m' *numbers of* $(e_1, e_2, \ldots e_m)$ *constitute an invariant nilpotent sub system containing every invariant nilpotent sub system of* $(e_1, e_2, \ldots e_m)$.

Let the system $(e_1, e_2, \ldots e_m)$ contain a nilpotent sub system $(C_1, C_2, \ldots C_p)$ such that

$$C_j e_i = \sum_{h=1}^{p} g_{ijh} C_h \qquad (i = 1, 2, \ldots m;\ j = 1, 2 \ldots p).$$

For $1 \leqq j \leqq p$, we then have, by *theorem I*,

$$S_1 C_j e_i = \sum_{h=1}^{p} g_{ijh} S_1 C_h = 0 \qquad (i = 1, 2, \ldots m);$$

therefore, $\Delta_1 = 0$, and thus $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system to which the sub system $(C_1, C_2, \ldots C_p)$ belongs. Similarly, we may show that, if the system $(e_1, e_2, \ldots e_m)$ contains a nilpotent sub system $(C_1, C_2, \ldots C_p)$ such that

$$e_i C_j = \sum_{h=1}^{p} g_{ijh} C_h \qquad (i = 1, 2 \ldots m;\ j = 1, 2 \ldots p),$$

it then contains an invariant nilpotent sub system which includes the sub system $(C_1, C_2, \ldots C_p)$.

If $(e_1, e_2, \ldots e_m)$ contains a sub system $(C_1, C_2, \ldots C_p)$ such that

$$S_1 C_1 = S_1 C_2 = \ldots = S_1 C_p = 0$$

or

$$S_2 C_1 = S_2 C_2 = \ldots = S_2 C_p = 0,$$

this sub system is nilpotent, since then, by *theorem I*, every number of the sub system is nilpotent. Thus, if

$$C = g_1 C_1 + g_2 C_2 + \ldots + g_p C_p$$

is any number of the sub system, we have

$$C^q = g_1^{(q)}C_1 + g_2^{(q)}C_2 + \dots + g_p^{(q)}C_p;$$

therefore,

$$S_1C^q = g_1^{(q)}S_1C_1 + g_2^{(q)}S_1C_2 + \dots + g_p^{(q)}S_1C_p = 0,$$

for any positive integer q.

§ 2.

For any given number

$$A = \sum_{i=1}^{m} a_i e_i$$

of the non-nilpotent system $(e_1, e_2, \dots e_m)$ there is a linear relation between $A, A^2, \dots A^{m+1}$; therefore, a smallest positive integer $\mu \leqq m + 1$ for which $A, A^2, \dots A^{\mu}$ are linearly related, and thus for which we have

$$(17) \qquad \Omega(A) \equiv A^{\mu} + p_1A^{\mu-1} + \dots + p_{\mu-1}A = 0,$$

where the p's are functions of the a's. Let $\rho_1, \rho_2, \dots \rho_r$, respectively of multiplicity $\mu_1, \mu_2, \dots \mu_r$, be the distinct non-zero roots, if any, of $\Omega(\rho) = 0$; when we have

$$(18) \qquad \Omega(\rho) \equiv \rho^{k_0}(\rho - \rho_1)^{k_1}(\rho - \rho_2)^{k_2} \dots (\rho - \rho_r)^{k_\rho},$$

where $k_0 \geqq 1$. Further, let

$$(19) \qquad W(\rho) \equiv \rho(\rho - \rho_1)(\rho - \rho_2) \dots (\rho - \rho_r).$$

Let now

$$f(A) = \sum_{h=1}^{p} c_h A^h$$

be any polynomial in A. If $f(A) = 0$, then $p \geqq \mu$ and $f(\rho)$ contains $\Omega(\rho)$; otherwise, there is a linear relation between $A, A^2, \dots A^{\mu-1}$, which is contrary to supposition. Wherefore, if $f(A)$ is nilpotent, $f(\rho)$ contains $W(\rho)$. Conversely, if $f(\rho)$ contains $W(\rho)$, $f(A)$ is nilpotent; and, if $f(\rho)$ contain $\Omega(\rho)$, then $f(A) = 0$.

Let A be non-nilpotent. Corresponding respectively to the $r \geqq 1$ distinct non-zero roots of $\Omega(\rho) = 0$, are r linearly independent numbers $I_1, I_2, \dots I_r$, linear in powers of A, which are severally idempotent and mutually nilfactorial: thus we have

$$(20) \quad I_uI_u = I_u \neq 0, \quad I_uI_v = 0 \qquad (u, v = 1, 2, \dots r;\ v \neq u).$$

If, for $1 \leqq u \leqq r$,

$$(21) \qquad \phi_o^{(u)}(\rho) \equiv \left(\frac{(\rho - \rho_u)^{k_u} - (o - \rho_u)^{k_u}}{-(o - \rho_u)^{k_u}}\right)^{k_o},$$

$$\phi_v^{(u)}(\rho) \equiv \left(\frac{(\rho - \rho_u)^{k_u} - (\rho_v - \rho_u)^{k_u}}{-(\rho_v - \rho_u)^{k_u}}\right)^{k_v}$$

$$(v = 1, 2, \dots u - 1, u + 1, \dots r),$$

and

$$(22) \quad f_u(\rho) \equiv \phi_o^{(u)}(\rho)\, \phi_1^{(u)}(\rho) \dots \phi_{u-1}^{(u)}(\rho)\, \phi_{u+1}^{(u)}(\rho) \dots \phi_r^{(u)}(\rho),$$

we may write

$$(23) \qquad I_u = f_u(A) \qquad (u = 1, 2, \dots r).$$[5]

I shall denote by $\bar{r}$ the greatest value of r for any number A of the system. Then $\bar{r}$ is the greatest number of idempotent numbers, mutually nilfactorial, contained in the system $(e_1, e_2, \dots e_m)$. For, if possible, let the system contain $p > \bar{r}$ numbers $K_1, K_2, \dots K_p$ satisfying the conditions

$$K^2_u = K_u \neq 0, \qquad K_u K_v = 0$$
$$(u, v = 1, 2, \dots p;\ v \neq u).$$

The K's are then linearly independent. If now

$$A = \lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_p K_p,$$

where the λ's are any p distinct scalars other than zero, the equation $\Omega(\rho) = 0$ has $p > \bar{r}$ distinct non-zero roots, which is contrary to supposition.

Let A be non-nilpotent and, for any positive integer p, let

$$(24) \qquad N^{(p)} \equiv \varpi_p(A) \equiv A^p - \sum_{u=1}^{r} \rho^p_u I_u;$$

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[5] For then, in the first place, $f_u(\rho)$ contains ρ as a factor; therefore, $f_u(A)$ is linear in powers of A. Moreover, for $1 \leqq u \leqq r$, $f_u(\rho)$ does not contain $\Omega(\rho)$, whereas $(f_u(\rho))^2 - f_u(\rho)$ does contain $\Omega(\rho)$; and, therefore, $I_u = f_u(A) \neq 0$, $I_u^2 - I_u = 0$. Further, for any two distinct integers u and v from 1 to r, $f_u(\rho) f_v(\rho)$ contains $\Omega(\rho)$; and, therefore, $I_u I_v = 0$. By the aid of the above two equations, we may show that $I_1, I_2, \dots I_r$ are linearly independent. Thus, if

$$J \equiv c_1 I_1 + c_2 I_2 + \dots + c_r I_r = 0,$$

then, for $\quad 1 \leqq u \leqq r, \quad c_u I_u = I_u J I_u = 0;$

and, therefore, $c_u = 0$.

in which case $N^{(p)}$ is nilpotent, since

$$\varpi_p(\rho) \equiv \rho^p - \sum_{u=1}^{r} \rho_u{}^p f_u(\rho)$$

contains $W(\rho)$: therefore, by *theorem I,*

$$(25) \qquad \begin{aligned} S_1 A^p &= \sum_{u=1}^{r} \rho_u{}^p S_1 I_u + S_1 N^{(p)} = \sum_{u=1}^{r} \rho_u{}^p S_1 I_u, \\ S_2 A^p &= \sum_{u=1}^{r} \rho_u{}^p S_2 I_u + S_2 N^{(p)} = \sum_{u=1}^{r} \rho_u{}^p S_2 I_u. \end{aligned}$$

If possible, let

$$S_1 A^p = S_1 A^{p+1} = \ldots = S_1 A^{p+r-1} = 0$$

for some positive integer p. By (25), we then have

$$\rho_1{}^{p+h} S_1 I_1 + \rho_2{}^{p+h} S_1 I_2 + \ldots + \rho_r{}^{p+h} S_1 I_r = 0$$
$$(h = 0, 1, 2, \ldots r - 1);$$

and since, by *theorem I,* neither $S_1 I_1$, $S_1 I_2$, $\ldots$ nor $S_1 I_r$ is zero, it follows that

$$\begin{vmatrix} \rho_1{}^p, \ldots \rho^{p+r-1} \\ \ldots\ldots\ldots\ldots \\ \rho_r{}^p, \ldots \rho_r{}^{p+r-1} \end{vmatrix} = 0,$$

which is impossible, since by supposition the ρ's are distinct and other than zero. *A fortiore,* we cannot have

$$S_1 A^p = S_1 A^{p+1} = \ldots = S_1 A^{p+\bar{r}-1} = 0$$

for any positive integer p. Similarly, we may show that we cannot have

$$S_2 A^p = S_2 A^{p+1} = \ldots = S_2 A^{p+\bar{r}-1} = 0$$

for any positive integer p if A is non-nilpotent.

We have now the following theorem.

Theorem III. *Let* $(e_1, e_2, \ldots e_m)$ *be any given non-nilpotent number system; and let* $\bar{r}$ *be the maximum number of idempotent numbers, mutually nilfactorial contained in the system. Then, if for any number*

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

of the system, we have, for some positive integer p,

$$S_1 A^{p+h} = 0 \qquad (h = 0, 1, 2, \dots r - 1),$$

or

$$S_2 A^{p+h} = 0 \qquad (h = 0, 1, 2, \dots r - 1),$$

A is nilpotent. Conversely, if A is nilpotent, these equations are all satisfied for any positive integer p.[6]

With respect to the idempotent numbers $I_1, I_2, \dots I_r$, linear in powers of any non-nilpotent number A, the number system may be *regularized* as follows. Let Γ_{oo} denote the aggregate of numbers

$$e_i - \sum_{u=1}^{r} I_u e_i - \sum_{u=1}^{r} e_i I_u + \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v$$

for $i = 1, 2, \dots m$. For any assigned integer u from 1 to r, let Γ_{uo} and Γ_{ou} denote, respectively, the aggregates

$$I_u e_i - \sum_{v=1}^{r} I_u e_i I_v \quad \text{and} \quad e_i I_u - \sum_{v=1}^{r} I_v e_i I_u$$

for $i = 1, 2, \dots m$; and, for any assigned pair of integers u, v from 1 to r, let Γ_{uv} denote the aggregate of numbers $I_u e_i I_v$ for $i = 1, 2, \dots m$. Further, for u and v any two integers from o to r, let m_{uv} denote the greatest number of linearly independent numbers of the aggregate Γ_{uv}; and, if $m_{uv} \neq 0$, let J_{uhv}, for $h = 1, 2, \dots m_{uv}$, denote any system of m_{uv} linearly independent numbers of Γ_{uv}. We then have, by (20),

$$(26) \qquad I_u J_{uhv} = J_{uhv} = J_{uhv} I_v, \qquad I_u J_{uh'o} = J_{uh'o} \qquad J_{oh''v} I_v = J_{oh''v}$$
$$(u, v = 1, 2, \dots r;\ h = 1, 2, \dots m_{uv};\ h' = 1, 2, \dots m_{uo};\ h'' = 1, 2, \dots m_{ov}),$$

$$(27) \qquad I_{u'} J_{uhv} = 0 = J_{uhv} I_{v'}$$
$$(u, v = 0, 1, 2, \dots r;\ h = 1, 2, \dots m_{uv};\ u'v' = 1, 2, \dots r;\ u' \neq u, v' \neq v).$$

We may now show that the J's are linearly independent. For, if

$$J \equiv \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{h=1}^{m_{pq}} g_{phq} J_{phq} + \sum_{p=1}^{r} \sum_{h=1}^{m_{po}} g_{pho} J_{pho}$$
$$+ \sum_{p=1}^{r} \sum_{h=1}^{m_{op}} g_{ohp} J_{ohp} + \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0,$$

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[6] Cf. paper by the author in the Trans. Am. Math. Soc., **5**, 545, note.

then, for any pair of integers u, v from 1 to r,

$$\sum_{h=1}^{m_{uv}} g_{uhv} J_{uhv} = I_u J I_v = 0;$$

and, since by supposition J_{u1v}, J_{u2v}, etc., are linearly independent, we have

$$g_{uhv} = 0 \qquad (u, v = 1, 2, \ldots r;\ h = 1, 2, \ldots m_{uv}).$$

Whence it follows that

$$J \equiv \sum_{p=1}^{r} \sum_{h=1}^{m_{po}} g_{pho} J_{pho} + \sum_{p=1}^{r} \sum_{h=1}^{m_{op}} g_{ohp} J_{ohp} + \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0;$$

and, therefore, for $1 \leqq u \leqq r$,

$$\sum_{h=1}^{m_{uo}} g_{uho} J_{uho} = I_u J = 0, \qquad \sum_{h=1}^{m_{ou}} g_{ohu} J_{ohu} = J I_u = 0.$$

From these equations we derive

$$g_{uho} = 0 \qquad (u = 1, 2, \ldots r;\ h = 1, 2, \ldots m_{uo}),$$

$$g_{ohu} = 0 \qquad (u = 1, 2, \ldots r;\ h = 1, 2, \ldots m_{ou}).$$

Thus, ultimately, we have

$$J \equiv \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0;$$

whence follows

$$g_{oho} = 0 \qquad (h = 1, 2, \ldots m_{oo}).$$

Since

$$(28) \quad e_i = \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v + \sum_{u=1}^{r} \left(I_u e_i - \sum_{v=1}^{r} I_u e_i I_v\right) + \sum_{u=1}^{r} \left(e_i I_u - \sum_{v=1}^{r} I_v e_i I_u\right) + \left(e_i - \sum_{u=1}^{r} I_u e_i - \sum_{u=1}^{r} e_i I_u + \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v\right) \qquad (i = 1, 2, \ldots m),$$

it follows that each unit of $(e_1, e_2, \ldots e_m)$, and thus that any number of this system, can be expressed linearly in terms of numbers in the $(r + 1)^2$ aggregates Γ_{uv} $(u, v = 0, 1, 2, \ldots r)$, and, therefore, linearly in terms of the J's. Whence it follows that we may take the J's as new units, and the number system thus transformed is *regularized* with respect to the idempotent numbers $I_1, I_2, \ldots I_r$.[7]

Since, for $1 \leqq u \leqq r$, I_u belongs to Γ_{uu}, we may put

$$(29) \qquad I_u = J_{u m_{uu} u} \qquad (u = 1, 2, \ldots r).$$

If now B' is any number of the system $(e_1, e_2, \ldots e_m)$ satisfying the equation $I_u B' = B'$, then, by (26) and (27),

$$B' = \sum_{v=0}^{r} \sum_{h=1}^{m_{uv}} b'_{vh} J_{uhv};$$

similarly, if $B'' I_u = B''$, we have

$$B'' = \sum_{v=0}^{r} \sum_{h=1}^{m_{vu}} b''_{vh} J_{vhu}.$$

Therefore, by *theorem I*,

$$(30) \qquad \begin{aligned} m S_1 I_u &= \sum_{v=0}^{r} m_{uv}, \\ m S_2 I_u &= \sum_{v=0}^{r} m_{vu}. \qquad (u = 1, 2, \ldots r). \end{aligned}$$

Let (u, v), for u, v any two integers from 0 to r denote a number of the aggregate Γ_{uv}. From (26) and (27), it then follows that the non-vanishing products of numbers in the several aggregates are given by the following equations:

$$(31) \qquad (u, v)\ (v, w) = (u, w) \qquad (u, v, w = 0, 1, 2, \ldots r);$$

and we further have

$$(32) \qquad (u, v)\ (v', w) = 0 \qquad (u, v, v', w = 0, 1, 2, \ldots r;\ v' \neq v).$$[8]

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[7] When the number system is thus transformed each of the new units is in one or other of Peirce's four "groups" or aggregates with respect to each of the r idempotent numbers $I_1, I_2, \ldots I_r$. Thus, if u is any integer from 1 to r and v, w any two integers from 0 to r other than u, then the units J_{uh_1u} $(1 \leqq h_1 \leqq m_{uu})$, J_{uh_2v} $(1 \leqq h_2 \leqq m_{uv})$, J_{vh_3u} $(1 \leqq h_3 \leqq m_{vu})$, and $J_{vh\,w}$ $(1 \leqq h_4 \leqq m)_{vw}$ are respectively in the first, second, third, and fourth groups with respect to I_u. See B. Peirce, loc. cit., p. 109.

We have now

$$m = \sum_{u=0}^{r} \sum_{v=0}^{r} m_{uv}.$$

[8] Cf. B. Peirce, loc. cit., p. 111.

Therefore, if in the square array,

$$\begin{array}{lllll}
\Gamma_{11}, & \Gamma_{12}, & \ldots & \Gamma_{1r}, & \Gamma_{10} \\
\Gamma_{21}, & \Gamma_{22}, & \ldots & \Gamma_{2r}, & \Gamma_{20} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\Gamma_{r1}, & \Gamma_{r2}, & \ldots & \Gamma_{rr}, & \Gamma_{r0} \\
\Gamma_{01}, & \Gamma_{02}, & \ldots & \Gamma_{0r}, & \Gamma_{00}
\end{array}$$

we strike out any p rows or any p columns, the units of the aggregates in the resulting array constitute a sub system of $(e_1, e_2, \ldots e_m)$. In particular, for $0 \leqq u \leqq r$, the units of Γ_{uu} constitute a sub system. Since, by (32), (u, v) is nilpotent if $u \neq v$, we have

$$(33) \quad S_1(u, v) = 0, \quad S_2(u, v) = 0 \qquad (u, v = 0, 1, 2, \ldots r;\ v \neq u).$$

Let now A be so chosen that $r = \bar{r}$, where, as above, $\bar{r}$ is the greatest value of r for any number A of the system. The units of Γ_{00} then constitute a nilpotent sub system; and, since every number of a nilpotent system or sub system is nilpotent, we now have

$$(34) \qquad S_1(0, 0) = 0, \qquad S_2(0\,0) = 0.$$

For, otherwise, if Γ_{00} contains an idempotent number I_0, we have

$$I_0 I_u = 0 = I_u I_0 \qquad (u = 1, 2, \ldots \bar{r})$$

by (27); and thus the number system $(e_1, e_2 \ldots e_m)$ contains $\bar{r} + 1$ idempotent numbers mutually nilfactorial, which is impossible, as shown above p. 19. Moreover, for $1 \leqq u \leqq \bar{r}$, there is now but one idempotent number in the aggregate Γ_{uu}. For, if possible, let Γ_{uu} contain a second idempotent number $I'_u = \sum\limits_{h=1}^{m_{uu}} c_h J_{uhu}$ other than I_u, in which case we have $I'^2_u = I'_u$; let

$$I''_u = I_u - I'_u,$$

when we have, by (20) and (26),

$$I''^2_u = I^2_u - I_u I'_u - I'_u I_u + I'^2_u = I_u - 2I'_u + I'_u = I''_u,$$

$$I'_u I''_u = I'_u(I_u - I'_u) = 0 = (I_u - I'_u) I'_u = I''_u I'_u,[9]$$

and, by (32),

$$I_v I''_u = I_v(I_u - I'_u) = 0 = (I_u - I'_u) I_v = I''_u I_v \quad (v \neq u).$$

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[9] Cf. B. Peirce, loc. cit., p. 112.

Wherefore, there are then at least $\bar{r} + 1$ idempotent numbers mutually nilfactorial, namely, I'_u, I''_u and I_v for $v = 1, 2, \ldots u - 1, u + 1, \ldots \bar{r}$, which is impossible.

The number system when regularized with respect to $\bar{r}$ idempotent numbers, so that Γ_{00} contains no idempotent number, and each of the aggregates Γ_{11}, Γ_{22}, ... $\Gamma_{\bar{r}\bar{r}}$ but a single idempotent number, is said to be *completely regularized.*

For $1 \leqq u \leqq \bar{r}$, we may now take the $m_{uu} - 1$ units other than I_u of the aggregate or system Γ_{uu} so that they shall all be nilpotent; in which case they constitute by themselves a nilpotent sub system, every number of which is, therefore, nilpotent.[10] I shall assume that in each of the aggregates Γ_{uu} $(u = 1, 2, \ldots \bar{r})$ the units have been so chosen.

Let

$$A = \sum_{u=0}^{\bar{r}} \sum_{v=0}^{\bar{r}} \sum_{h=1}^{m_{uv}} a_{uhv} J_{uhv}. \tag{35}$$

By equations (29), (30), (33), and (34), and by what has just been stated, we now have

$$\begin{aligned} S_1 A &= \sum_{u=1}^{\bar{r}} a_{um_{uu}u} S_1 I_u \\ &= \frac{1}{m} \sum_{u=1}^{\bar{r}} \sum_{v=0}^{\bar{r}} a_{um_{uu}u} m_{uv}, \end{aligned} \tag{36}$$

$$\begin{aligned} S_2 A &= \sum_{u=1}^{\bar{r}} a_{um_{uu}u} S_2 I_u \\ &= \frac{1}{m} \sum_{u=1}^{\bar{r}} \sum_{v=0}^{\bar{r}} a_{um_{uu}u} m_{vu}. \end{aligned} \tag{37}$$

I shall say that the two idempotent units I_u and I_v $(1 \leqq u \leqq \bar{r}$, $1 \leqq v \leqq \bar{r}$, $v \neq u)$ are *connected* if there are two numbers $(u, v)'$ and $(v, u)'$ such that

$$S_1\,(u, v)'\,(v, u)' \neq 0;$$

otherwise, *not connected.* If I_u and I_v are not connected, then

$$S_1\,(u, v)\,(v, u) = 0$$

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[10] This theorem is due to B. Peirce, loc. cit., p. 118. His proof is defective. The first proof, I believe, of the theorem without the aid of the theory of groups was given by me in the Transactions American Mathematical Society, **5**, p. 547, by employing the generalized scalar function.

for any two numbers (u, v) of Γ_{uv} and (v, u) of Γ_{vu}. Let $(u, v)'$, $(v, u)'$ be any two numbers of Γ_{uv}, Γ_{vu} respectively. Then

$$(u, v)' (v, u)' = \rho I_u - N_u$$

by (31), where N_u is linear in the nilpotent units of Γ_{uu} and is, therefore, either zero or nilpotent, and thus $N_u^{p+1} = 0$ for some positive integer p. Furthermore,

$$S_1 (u, v)' (v, u)' = \rho S_1 I_u.^{11}$$

If now I_u and I_v are connected, then, for a proper choice of $(u, v)'$, $(v, u)'$, we have $S_1 (u, v)' (v, u)' \neq 0$, in which case $\rho \neq 0$: therefore, we may put

$$(v, u)'' = \frac{1}{\rho^{p+1}} (v, u)' (\rho^p I_u + \rho^{p-1} N_u + \ldots + \rho N_u^{p-1} + N_u^p),$$

when we have

$$\begin{aligned}(u, v)' (v, u)'' &= \frac{1}{\rho^{p+1}} (\rho I_u - N_u) (\rho^p I_u + \rho^{p-1} N_u + \ldots + \rho N_u^{p-1} + N_u^p) \\ &= \frac{1}{\rho^{p+1}} (\rho^{p+1} I_u - N_u^{p+1}) = I_u;\end{aligned}$$

and since

$$\begin{aligned}[(v, u)'' (u, v)']^2 &= (v, u)''. (u, v)' (v, u)''. (u, v)' \\ &= (v, u)'' I_u (u, v)' = (v, u)'' (u, v)',\end{aligned}$$

it follows that

$$(v, u)'' (u, v)' = I_v,$$

otherwise, there is more than one idempotent unit in Γ_{vv}, which is contrary to supposition. Wherefore, *if I_u and I_v are connected, there are two numbers $(u, v)'$ and $(v, u)'$*, of Γ_{uv} and Γ_{vu} respectively, such that

$$(u, v)' (v, u)' = I_u, \qquad (v, u)' (u, v)' = I_v;$$

and conversely, since in this case

$$S_1 (u, v)' (v, u)' = S_1 I_u \neq 0.$$

If I_u and I_v are connected, and I_v and I_w are also connected, then I_u and I_w are connected, where u, v, w are any three distinct integers from

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[11] Further, $S_2 (u, v)' (v, u)' = \rho S_2 I_u$; therefore, if $S_1 (u, v)' (v, u)' \neq 0$, then $S_2 (u, v)' (v, u)' \neq 0$, and conversely.

1 to r. For, in this case there are two pairs of numbers, namely, $(u, v)'$, $(v, u)'$ and $(v, w)'$, $(w, v)'$ such that

$$(u, v)'\,(v, u)' = I_u, \qquad (v, u)'\,(u, v)' = I_v,$$
$$(v, w)'\,(w, v)' = I_v, \qquad (w, v)'\,(v, w)' = I_w.$$

Therefore, if

$$(u, w)' = (u, v)'\,(v, w)', \qquad (w, u)' = (w, v)'\,(v, u)',$$

we have, by (26),

$$\begin{aligned}(u, w)'\,(w, u)' &= (u, v)'.\,(v, w)'\,(w, v)'.\,(v, u)'\\ &= (u, v)'\,I_v\,(v, u)' = (u, v)'\,(v, u)' = I_u,\\ (w, u)'\,(u, w)' &= (w, v)'.\,(v, u)'\,(u, v)'.\,(v, w)'\\ &= (w, v)'\,I_v\,(v, w)' = (w, v)'\,(v, w)' = I_w.\end{aligned}$$

For u, v any two distinct integers from 1 to r, let I_u and I_v be connected. Thus let

$$(u, v)'\,(v, u)' = I_u, \qquad (v, u)'\,(u, v)' = I_v.$$

Let $k = m_{uu} - 1$; and let the nilpotent units of Γ_{uu} be denoted by $N_u^{(1)}, N_u^{(2)}, \ldots N_u^{(k)}$. Then $(u, v)'$ and the products $N_u^{(h)} \cdot (u, v)'$, for $h = 1, 2, \ldots k$, are numbers of the aggregate Γ_{uv} linearly independent. For, if

$$g_0 \cdot (u, v)' + \sum_{h=1}^{k} g_h N_u^{(h)} \cdot (u, v)' = 0,$$

then

$$g_0 I_u + \sum_{h=1}^{k} g_h N_u^{(h)} = [g_0 \cdot (u, v)' + \sum_{h=1}^{k} g_h N_u^{(h)} \cdot (u, v)']\,(v, u)' = 0$$

which is impossible, unless the g's are all zero. Therefore,

$$m_{uv} \geqq k + 1 = m_{uu}.$$

Moreover, there is no number in the aggregate Γ_{uv} linearly independent of these $k + 1$ numbers of this aggregate. For, if (u, v) is any number of this aggregate, since $(u\,v)\,(v, u)'$ belongs to the aggregate Γ_{uu}, we have

$$(u, v)\,(v, u)' = c_0 I_u + \sum_{h=1}^{k} c_h N_u^{(h)};$$

and, therefore,

$$\begin{aligned}(u, v) &= (u, v)\, I_v = (u, v) \cdot (v, u)'\, (u, v)' \\ &= (c_0 I_u + \sum_{h=1}^{k} c_h N_u^{(h)})\, (u, v)' \\ &= c_0\, (u,v)' + \sum_{h=1}^{k} c_h\, N_u^{(h)}\, (u, v)'.\end{aligned}$$

Whence it follows that m_{uv} cannot exceed $m_{uu} = k + 1$; and, therefore, $m_{uv} = m_{uu}$. Similarly, we may show next that $(v, u)'$ and the product $(v, u)' N_u^{(h)}$, for $h = 1, 2, \ldots k$, are linearly independent, and that in terms of these numbers every number of the aggregate Γ_{vu} can be expressed linearly. Finally, that I_v and the k products $(v, u)' N_u^{(h)} (u, v)'$, for $h = 1, 2, \ldots k$, are linearly independent, and that in terms of these numbers every number of the aggregate Γ_{vv} can be expressed linearly. Therefore, in particular, *if* I_u *and* I_v *are connected,*

$$m_{uu} = m_{uv} = m_{vu} = m_{vv}.$$

For $1 \leqq i \leqq m$ and u, v any two integers from 0 to $\bar{r}$, let $(u, v)_i$ denote the component of e_i in Γ_{uv}. We then have

$$e_i = \sum_{p=0}^{\bar{r}} \sum_{q=0}^{\bar{r}} (p, q)_i \qquad (i = 1, 2, \ldots m). \tag{38}$$

Whence, from (32), we derive

$$\begin{aligned}S_1 (u, v)\, e_i &= \sum_{p=0}^{\bar{r}} \sum_{q=0}^{\bar{r}} S_1 (u, v)\, (p, q)_i \\ &= \sum_{q=0}^{\bar{r}} S_1 (u, v)\, (v, q)_i \\ &= \sum_{q=0}^{\bar{r}} S_1 (v, q)_i\, (u, v) = S_1 (v, u)_i\, (u, v) = S_1 (u, v)\, (v, u)_i \\ &\qquad (u, v = 0, 1, 2 \ldots \bar{r};\ i = 1, 2, \ldots m).\end{aligned} \tag{39}$$

We may now show first that if, for $0 \leqq u \leqq \bar{r}$, the aggregate Γ_{0u} contains any unit, that is, if $m_{u0} > 0$, the number system $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system. For, let $(u, o) \neq 0$, and let

$$(o, u)_i (u, o) = (o, o)_i' \qquad (i = 1, 2, \ldots m);$$

when, by (34) and (39), we have

$$\begin{aligned} S_1(u, o)\, e_i &= S_1(u, o)\, (o, u)_i \\ &= S_1(o, u)_i\, (u, o) = S_1(o, o)_i' = 0 \\ &\qquad (i = 1, 2, \ldots m), \end{aligned}$$

and thus (u, o) satisfies equations (6). Similarly, if $m_{ou} > 0\, (1 \leqq u \leqq r)$, we may show that $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system.

Again, if $\Gamma_{uu}\, (1 \leqq u \leqq r)$ contains more than one unit, that is, if $m_{uu} > 1$, the system $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system. For, in this case, there is a nilpotent number (u, u) of Γ_{uu} whose product with any number of this aggregate is, therefore, nilpotent;[12] and thus $(u\, u)\ (u, u)_i$, for $i = 1, 2, \ldots m$, is nilpotent: therefore,

$$S_1(u, u)\, e_i = S_1(u, u)\, (u, u)_i = 0 \qquad (i = 1, 2, \ldots m),$$

and thus (u, u) is a solution of equations (6). If, for u, v any two distinct integers from 1 to r, I_u and I_v are connected, and either Γ_{uv} or Γ_{vu} contains more than one unit; that is, if either $m_{uv} > 1$ or $m_{vu} > 1$, the system $(e_1, e_2, \ldots e_m)$ contains a nilpotent sub system. For then, by the theorem p. 645, we have $m_{uu} > 1$. Further, if I_u and I_v are not connected, and either Γ_{uv} or Γ_{vu} contains one or more units, that is, if $m_{uv} > 0$ or $m_{vu} > 0$, the number system contains an invariant nilpotent sub system. For let $(u, v) \neq 0$: in this case, by the theorem given, p. 642, we have

$$S_1(u, v)\, (v, u)_i = 0 \qquad (i = 1, 2, \ldots m);$$

therefore,

$$S_1(u, v)\, e_i = S_1(u, v)\, (v, u)_i = 0 \qquad (i = 1, 2, \ldots m),$$

and thus (u, v) satisfies equations (6). Finally, if I_u and I_v are not connected and $m_{vu} > 0$, $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system.

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[12] Namely, when $m_{uu} > 1$, any number (u, u) linear in the nilpotent units of Γ_{uu} is such a number. For since I_u is a modulus of the system Γ_{uu}, these nilpotent units constitute an invariant nilpotent sub system of Γ_{uu}. Wherefore, the products of (u, u) and any number of Γ_{uu} belongs to this nilpotent sub system, and is, therefore, nilpotent.

I shall now assume that the number system $(e_1, e_1, \dots e_m)$ contains no invariant nilpotent sub system, in which case, by what has just been proved, we have

$$(40) \qquad m_{uo} = m_{ou} = m_{oo} = 0 \qquad (u = 1, 2, \dots \bar{r}),$$

that is, no number of the system is contained in Γ_{oo} nor in either of the aggregates Γ_{uo}, Γ_{ou} for $u = 1, 2, \dots \bar{r}$. Further,

$$(41) \qquad m_{uu} = 1 \qquad (u = 1, 2, \dots \bar{r}),$$

that is, I_u is the only unit in Γ_{uu} for $1 \leqq u \leqq \bar{r}$. Finally, for u and v any two distinct integers from 1 to $\bar{r}$, if I_u and I_v are connected,

$$m_{uv} = m_{vu} = 1;$$

whereas, if I_u and I_v are not connected,

$$m_{uv} = m_{vu} = 0.$$

In the present case, the number system contains a modulus, viz.,

$$(42) \qquad \epsilon = I_1 + I_2 + \dots + I_{\bar{r}},$$

since, for u, v any two integers from 1 to $\bar{r}$, if Γ_{uv} contains a unit J_{u1v}, we have

$$\epsilon J_{u1v} = J_{u1v} = J_{u1v}\epsilon$$

by (26) and (27).

It is, with the present assumption, convenient to modify our notation to indicate the connection which may exist between certain of the idempotent numbers, $I_1, I_2, \dots I_{\bar{r}}$. I shall, therefore, suppose these numbers arranged in ν aggregates, $1 \leqq \nu \leqq \bar{r}$, containing respectively $\mu_1, \mu_2, \dots \mu_\nu$ of the I's, where $\sum_{p=1}^{\nu} \mu_p = \bar{r}$, any two idempotent numbers in the same aggregate being connected, but no pair of idempotent numbers in different aggregates being connected; and, for $1 \leqq p \leqq \nu$, I shall denote by $I_u^{(p)}$ $(u = 1, 2, \dots \mu_p)$ the idempotent numbers in the p^{th} aggregate. The $\bar{r}^2$ aggregates of numbers, formerly denoted by $\Gamma_{u,v}$ for $u, v = 1, 2, \dots \bar{r}$, into, one or other of which the units fall when the system is regularized as above and contains no invariant nilpotent sub system, will now be denoted by $\Gamma_{u,v}^{(p,q)}$ for $p, q = 1, 2, \dots \nu$, and for $u = 1, 2, \dots \mu_p$ and $v = 1, 2, \dots \mu_q$;

and the number of linearly independent numbers in $\Gamma_{u,v}{}^{(p,q)}$ will be denoted by $m_{u,v}{}^{(p,q)}$[13]. By what is shown above we now have

$$(44) \qquad m_{uv}{}^{(p,p)} = 1 \qquad (p = 1, 2, \dots \nu;\ u, v = 1, 2, \dots \mu_p),$$

$$(45) \qquad m_{uv}{}^{(p,q)} = 0$$
$$(p, q = 1, 2, \dots \nu;\ q \neq p;\ u = 1, 2, \dots \mu_p;\ v = 1, 2, \dots \mu_q)^{14}.$$

For $1 \leqq p \leqq \nu$ and u and v any two distinct integers from 1 to μ_p, we may now, in harmony with the preceding notation, denote the single unit of $\Gamma_{uv}{}^{(p,p)}$ by $J_{uv}{}^{(p)}$; and if, further, we denote by $J_{uu}{}^{(p)}$ the idempotent unit $I_u{}^{(p)}$ of $\Gamma_{uu}{}^{(p,p)}$, we shall have as the multiplication table of the system

$$(46) \qquad J_{uv}{}^{(p)} J_{vw}{}^{(p)} = \rho_{uvw} J_{uw}{}^{(p)}, \qquad J_{uv}{}^{(p)} J_{v'w}{}^{(p)} = 0$$
$$(p = 1, 2, \dots \nu;\ u, v, v', w = 1, 2, \dots \mu_p;\ v' \neq v),$$

$$(47) \qquad J_{uv}{}^{(p)} J_{u'v'}{}^{(q)} = 0$$
$$(p, q = 1, 2, \dots \nu;\ q \neq p;\ u,v = 1, 2, \dots \mu_p;\ u', v', = 1, 2, \dots \mu_q)$$

by (31), (32), and (44), where $\rho_{uuv} = \rho_{uvv} = 1$. For $1 \leqq p \leqq \nu$, and for u, v any two integers from 1 to μ_p, it follows from (44) that

$$\rho_{uvu}{}^{(p)} S_1 J_{uu}{}^{(p)} = S_1 J_{uv}{}^{(p)} J_{vu}{}^{(p)} \neq 0,$$

and thus $\rho_{uvu}{}^{(p)} \neq 0$, otherwise $J_{uu}{}^{(p)} = I_u{}^{(p)}$ and $J_{vv}{}^{(p)} = I_v{}^{(p)}$ are not connected; and, since

$$\begin{aligned}(\rho_{uvu}{}^{(p)})^2 J_{uu}{}^{(p)} &= (\rho_{uvu}{}^{(p)} J_{uu}{}^{(p)})^2 = (J_{uv}{}^{(p)} J_{vu}{}^{(p)})^2 \\ &= J_{uv}{}^{(p)} \cdot J_{vu}{}^{(p)} J_{uv}{}^{(p)} \cdot J_{vu}{}^{(p)} = \rho_{vuv}{}^{(p)} J_{uv}{}^{(p)} J_{vv}{}^{(p)} J_{vu}{}^{(p)} \\ &= \rho_{vuv}{}^{(p)} J_{uv}{}^{(p)} J_{vu}{}^{(p)} = \rho_{vuv}{}^{(p)} \rho_{uvu}{}^{(p)} J_{uu}{}^{(p)} \neq 0,\end{aligned}$$

we have $\rho_{uvu}{}^{(p)} = \rho_{vuv}{}^{(p)}$. Further, for $1 \leqq w \leqq \mu_p$,

$$\begin{aligned}\rho_{uvw}{}^{(p)} \rho_{vuw}{}^{(p)} J_{vw} &= \rho_{uvw}{}^{(p)} J_{vu}{}^{(p)} J_{uw}{}^{(p)} = J_{vu}{}^{(p)} \cdot J_{uv}{}^{(p)} J_{vw}{}^{(p)} \\ &= J_{vu}{}^{(p)} J_{uv}{}^{(p)} \cdot J_{vw}{}^{(p)} = \rho_{vuv}{}^{(p)} J_{vv}{}^{(p)} J_{vw}{}^{(p)} \\ &= \rho_{vuv}{}^{(p)} J_{vw}{}^{(p)} \neq 0;\end{aligned}$$

and, therefore, $\rho_{uvw}{}^{(p)} \neq 0$.

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[13] Thus, whereas, formerly Γ_{uv} denoted the aggregate of numbers $I_u e_i I_v$ for $i = 1, 2, \dots m$, of which m_{uv} were linearly independent, $\Gamma_{uv}{}^{(p,q)}$ is now the aggregate of numbers $I_u{}^{(p)} e_i I_v{}^{(q)}$ for $i = 1, 2, \dots m$, of which $m_{uv}{}^{(p,q)}$ are linearly independent.

[14] Therefore,

$$m = \sum_{p=1}^{\nu} \sum_{q=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_q} m_{uv}{}^{(p,q)} = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} m_{uv}{}^{(p,p)} = \sum_{p=1}^{\nu} \mu_p{}^2.$$

Let

$$(48)\qquad \bar{J}_{uv}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}\rho_{1v1}}} J_{u1}^{(p)} J_{1v}^{(p)}$$

$$(p = 1, 2, \ldots \nu;\ u, v = 1, 2, \ldots \mu_p).$$

Then

$$(49)\qquad J_{uv}^{(p)} = \frac{1}{\rho_{u1v}^{(p)}} J_{u1}^{(p)} J_{1v}^{(p)} = \frac{\sqrt{\rho_{1u1}^{(p)}\rho_{1v1}^{(p)}}}{\rho_{u1v}^{(p)}} \bar{J}_{uv}^{(p)}$$

$$(p = 1, 2, \ldots \nu;\ u, v = 1, 2, \ldots \mu_p)$$

by (46); and, therefore, we may take the $\bar{J}$'s as new units. We now have

$$(50)\qquad \bar{J}_{uv}^{(p)} \bar{J}_{vw}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}^{(p)}\rho_{1v1}^{(p)} \cdot \rho_{1v1}^{(p)}\rho_{1w1}^{(p)}}} J_{u1}^{(p)} J_{1v}^{(p)} \cdot J_{v1}^{(p)} J_{1w}^{(p)}$$

$$= \frac{1}{\sqrt{\rho_{1u1}^{(p)}\rho_{1w1}^{(p)}}} J_{u1}^{(p)} J_{1w}^{(p)} \text{[15]} = \bar{J}_{uw}^{(p)},$$

$$\bar{J}_{uv}^{(p)} \bar{J}_{v'w}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}^{(p)}\rho_{1v1}^{(p)} \cdot \rho_{1v'1}^{(p)}\rho_{1w1}^{(p)}}} J_{u1}^{(p)} J_{1v}^{(p)} \cdot J_{v'1}^{(p)} J_{1w}^{(p)} = 0$$

$$(p = 1, 2, \ldots \nu;\ u, v, v', w = 1, 2, \ldots \mu_p;\ v' \neq v),$$

$$(51)\ \bar{J}_{uv}^{(p)} \bar{J}_{u'v'}^{(q)} = \frac{1}{\sqrt{\rho_{1u1}^{(p)}\rho_{1v1}^{(p)} \cdot \rho_{1u'1}^{(q)}\rho_{1v'1}^{(q)}}} J_{u1}^{(p)} J_{1v}^{(p)} \cdot J_{u'1}^{(q)} J_{1v'}^{(q)} = 0$$

$$(p, q = 1, 2, \ldots \nu;\ q \neq p;\ u, v = 1, 2, \ldots \mu_p;\ u', v' = 1, 2, \ldots \mu_q).$$

For $1 \leqq p \leqq \nu$, the units $\bar{J}_{uv}^{(p)}$ for $u, v = 1, 2, \ldots \mu_p$ constitute a quadrate of order μ_p; and, therefore, in the present case, the number system is constituted by ν mutually nilfactorial quadrates.[16] For the modulus ϵ of the system we now have

$$(52)\qquad \epsilon = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} I_u^{(p)} = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \bar{J}_{uu}^{(p)}$$

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[15] For

$$J_{u1}^{(p)} J_{1v}^{(p)} \cdot J_{v1}^{(p)} J_{1w}^{(p)} = J_{u1}^{(p)} \cdot J_{1v}^{(p)} J_{v1}^{(p)} \cdot J_{1w}^{(p)} = \rho_{1v1}^{(p)} J_{u1}^{(p)} J_{11}^{(p)} J_{1w}^{(p)}$$
$$= \rho_{1v1}^{(p)} J_{u1}^{(p)} J_{1w}^{(p)}.$$

[16] A *quadrate* is a hyper complex number system with $m = \bar{m}^2$ units ϵ_{uv} $(u, v = 1, 2, \ldots \bar{m})$ which can be so chosen that

$$\epsilon_{uv}\epsilon_{vw} = \epsilon_{uw}, \qquad \epsilon_{uv}\epsilon_{v'w} = 0 \qquad (u, v, v', w = 1, 2, \ldots \bar{m};\ v' \neq v).$$

B. Peirce, Am. Journ. Maths., 4, 217.

By (30), (33), (40), (44), and (45), we now have

$$(53)\qquad m S_1 \bar{J}_{uu}^{(p)} = \sum_{q=1}^{\nu} \sum_{v=1}^{\mu_q} m_{uv}^{(p,q)} = \sum_{v=1}^{\mu_p} m_{uv}^{(p,p)} = \mu_p$$

$$= \sum_{v=1}^{\mu_p} m_{vu}^{(p,p)} = \sum_{q=1}^{\nu} \sum_{v=1}^{\mu_q} m_{vu}^{(q,p)} = m S_2 \bar{J}_{uu}^{(p)}$$

$$(p = 1, 2, \ldots \nu;\ u = 1, 2, \ldots \mu_p),$$

$$(54)\qquad S_1 \bar{J}_{uv}^{(p)} = 0, \qquad S_2 \bar{J}_{uv}^{(p)} = 0$$

$$(p = 1, 2, \ldots \nu;\ u, v = 1, 2, \ldots \mu_p;\ v \neq u).$$

And since, for any number $A = \sum_{i=1}^{m} a_i e_i$, we may now put

$$(55)\qquad A = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} \bar{J}_{uv}^{(p)},$$

we have

$$(56)\qquad S_1 A = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} S_1 \bar{J}_{uv}^{(p)}$$

$$= \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} S_2 \bar{J}_{uv}^{(p)} = S_2 A.$$

Therefore, in particular,

$$(57)\qquad S_1 e_i e_j = S_2 e_i e_j \qquad (i, j = 1, 2, \ldots m),$$

and thus we have $\Delta_1 = \Delta_2$ also in the case now considered, when the system $(e_1, e_2, \ldots e_m)$ contains no invariant nilpotent sub system and neither Δ_1 nor Δ_2 is zero.

From the conditions, necessary and sufficient, that the m^3 constants γ_{ijk} $(i, j, k = 1, 2, \ldots m)$ shall constitute the constants of multiplication of a hyper complex number system in m units, viz.,

$$(58)\qquad \sum_{k=1}^{m} \gamma_{ijk} \gamma_{khl} = \sum_{k=1}^{m} \gamma_{ikl} \gamma_{jhk}$$

$$(i, j, h, l = 1, 2, \ldots m),$$

we derive

$$(59) \quad m^m \Delta_1 = \begin{vmatrix} m S_1 (e_i e_j) \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma_h \Sigma_k \gamma_{ijk} \gamma_{khh} \\ (i, j = 1, 2, \dots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h \Sigma_k \gamma_{ikh} \gamma_{jhk} \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

$$= \begin{vmatrix} \gamma_{i11}, \dots \gamma_{i1m}, \dots \gamma_{im1}, \dots \gamma_{imm} \\ (i = 1, 2, \dots m) \end{vmatrix} \begin{vmatrix} \gamma_{j11}, \dots \gamma_{jm1}, \dots \gamma_{j1m}, \dots \gamma_{jmm} \\ (j = 1, 2, \dots m) \end{vmatrix},$$

$$(60) \quad m^m \Delta_2 = \begin{vmatrix} m S_2 e_j e_i \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma_h \Sigma_k \gamma_{jik} \gamma_{hkh} \\ (i, j = 1, 2, \dots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h \Sigma_k \gamma_{hjk} \gamma_{kih} \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

$$= \begin{vmatrix} \gamma_{1i1}, \dots \gamma_{1im}, \dots \gamma_{mi1}, \dots \gamma_{mim} \\ (i = 1, 2, \dots m) \end{vmatrix} \begin{vmatrix} \gamma_{1j1}, \dots \gamma_{mj1}, \dots \gamma_{1jm}, \dots \gamma_{mjm} \\ (j = 1, 2, \dots m) \end{vmatrix}.$$

A number system containing no invariant sub system is termed by Cartan a *simple system* (*systeme simple*), and he shows that such a system is what is here termed a quadrate. A non-simple system containing no invariant nilpotent sub system Carten terms *semi-simple.*[17] Such a system is constituted by nilfactorial quadrates of which the invariant sub systems are any $p (1 \leqq p < \nu)$ of these quadrates. By what is shown above it appears that $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ is the condition necessary and sufficient that a number system shall be either simple or semi-simple. We have, therefore, the following theorem:

Theorem IV. *Let* $e_1, e_2, \dots e_m$ *be the units of any hyper complex number system, and let*

$$\Delta_1 \equiv \begin{vmatrix} S_1 e_i e_j \\ (i, j = 1, 2, \dots r) \end{vmatrix}, \qquad \Delta_2 \equiv \begin{vmatrix} S_2 e_i e_j \\ (i, j = 1, 2, \dots r) \end{vmatrix}$$

Then $\Delta_1 = \Delta_2$. *If* $\Delta_1 \neq 0$, *the number system contains a modulus and is either simple or semi-simple, that is, is constituted by* $\nu \geqq 1$ *mutually nilfactorial quadrates; and, conversely, in this case,* $\Delta_1 = \Delta_2 \neq 0$.

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[17] Comptes Rendus, **124,** 1218 (1897).

§ 3.

It has been shown by C. S. Peirce that any given hyper complex number system $(e_1, e_2, \ldots e_m)$ is a sub system of a quadrate of order n, where the greatest value n need assume is $m + 1$. This is, of course equivalent to the theorem that any given number system can be represented by a matrix whose order need not exceed $m + 1$.[18]

Let now $(e_1, e_2, \ldots e_m)$ be any given number system; let e_{uv} $(u, v = 1, 2, \ldots n)$ be the units of the quadrate of which $(e_1, e_2, \ldots e_m)$ is a sub system, when we have

$$\epsilon_{uv}\epsilon_{vw} = \epsilon_{uw}, \qquad \epsilon_{uv}\epsilon_{v'w} = 0 \qquad (61)$$
$$(u, v, v', w = 1, 2, \ldots n;\ v' \neq v);$$

and let

$$e_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \theta_{uv}^{(i)} \epsilon_{uv} \qquad (i = 1, 2, \ldots m). \qquad (62)$$

The units $e_1, e_2, \ldots e_m$ may then be regarded as represented, respectively, by the m linearly independent matrices $E_1, E_2, \ldots E_m$, where E_i, for $i = i, 2, \ldots m$, is defined by the system of equations,

$$(\xi_1', \xi_2', \ldots \xi_n') = \left(\begin{array}{|l|} \theta_{11}^{(i)}, \theta_{12}^{(i)}, \ldots \theta_{1n}^{(i)} \\ \theta_{21}^{(i)}, \theta_{22}^{(i)}, \ldots \theta_{2n}^{(i)} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \theta_{n1}^{(i)}, \theta_{n2}^{(i)}, \ldots \theta_{nn}^{(i)} \end{array}\right)(\xi_1, \xi_2, \ldots \xi_n), \qquad (63)$$

and any number $x = \sum_{i=1}^{m} x_i e_i$ of $(e_1, e_2, \ldots e_m)$ by the matrix of the linear substitution

(64)

$$(\xi_1', \xi_2', \ldots \xi_n') = \left(\begin{array}{|l|} \sum_{i=1}^{m} x_i\theta_{11}^{(i)}, \sum_{i=1}^{m} x_i\theta_{12}^{(i)}, \ldots \sum_{i=1}^{m} x_i\theta_{1n}^{(i)} \\ \sum_{i=1}^{m} x_i\theta_{21}^{(i)}, \sum_{i=1}^{m} x_i\theta_{22}^{(i)}, \ldots \sum_{i=1}^{m} x_i\theta_{2n}^{(i)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum_{i=1}^{m} x_i\theta_{n1}^{(i)}, \sum_{i=1}^{m} x_i\theta_{n2}^{(i)}, \ldots \sum_{i=1}^{m} x_i\theta_{nn}^{(i)} \end{array}\right)(\xi_1, \xi_2, \ldots \xi_n).$$

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[18] Loc. cit., p. 221; also These Proceedings, **10,** 392 (1875). In certain cases, as shown by Peirce, we may take $n \leqq m$; in other cases, n must be greater than m. See § 4.

For any number of the quadrate $\epsilon_{uv}(u,\ v = 1,\ 2,\ \dots n)$ the two scalar functions with respect to this number system defined in *theorem I* are equal as shown in § 2; and, therefore, but a single symbol is required for these functions. I shall denote by $\overline{S}\,\overline{A}$ the two equal scalar functions of any number

$$(65)\qquad \overline{A} = \sum_{u=1}^{n}\sum_{v=1}^{n} a_{uv}\epsilon_{uv} = \begin{vmatrix} a_{11}, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots & a_{nn} \end{vmatrix}$$

of the quadrate; and, by *theorem I,* we then have

$$(66)\qquad \overline{S}\epsilon_{uu} = \frac{1}{n},\ \overline{S}\epsilon_{uv} = 0 \qquad (u,\ v = 1,\ 2,\ \dots n;\ v \neq u),^{19}$$

and, therefore,

$$(67)\qquad \overline{S}\,\overline{A} = \sum_{u=1}^{n}\sum_{v=1}^{n} a_{uv}\overline{S}\epsilon_{uv} = \frac{1}{n}\sum_{u=1}^{n} a_{uu}.$$

I shall denote simply by 1 the modulus of the quadrate, and $\rho 1$, for any scalar ρ, simply by ρ. We have

$$(68)\qquad 1 = \sum_{u=1}^{n} \epsilon_{uu}.$$

Any number $\overline{A} = \sum_{u=1}^{n}\sum_{v=1}^{n} a_{uv}\epsilon_{uv}$ of the quadrate satisfies an equation

$$(69)\qquad \phi(A) \equiv (\overline{A} - \rho_1)(\overline{A} - \rho_2)\ \dots\ (\overline{A} - \rho_n) = 0,$$

where the ρ's are scalars; and we have

$$(70)\quad \phi(\rho) \equiv \begin{vmatrix} \rho - a_{11}, & -a_{12}, & \dots & -a_{1n} \\ -a_{21}, & \rho - a_{22}, & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1}, & -a_{n2}, & \dots & \rho - a_{nn} \end{vmatrix} \equiv (\rho - \rho_1)(\rho - \rho_2)\ \dots\ (\rho - \rho_n).^{20}$$

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[19] For the number of linearly independent numbers $\overline{X}$ of the quadrate satisfying the equation $e_{uu}\overline{X} = \overline{X}$ is n, since every such number is linearly expressible in $\epsilon_{u1},\ \epsilon_{u2},\ \dots\ \epsilon_{un}$, and each of these numbers satisfies this equation. Therefore, by *theorem I,* $n^2\overline{S}_1\epsilon_{un} = n$. Similarly, the number of linearly independent numbers $\overline{X}$ of the quadrate satisfying the equation $\overline{X}\epsilon_{uu} = \overline{X}$ is also n; and, therefore, $n^2\overline{S}_2\epsilon_{uu} = n$. Since, for $v \neq u$, ϵ_{uv} is nilpotent, $\overline{S}_1\epsilon_{uv} = \overline{S}_2\epsilon_{uv} = 0 \quad (v \neq u)$.

[20] Cayley: Philosophical Transactions, p. 800 (1858).

The polynomial $\phi(\rho)$ is termed the "characteristic function" of $\bar{A}$, and $\phi(\rho) = 0$ the "characteristic equation" of $\bar{A}$. Since, by (67), $n\,\bar{S}\,\bar{A}$ is the sum of the constituents in the principal diagonal of the matrix representing $\bar{A}$, it follows that *$n\,\bar{S}\,\bar{A}$ is equal to the sum of the roots of the characteristic equation of $\bar{A}$.*

If $\bar{A}$ is idempotent, the roots of its characteristic equation are 0 and 1. Wherefore, *if $\bar{A}$ is idempotent, $n\,\bar{S}\,\bar{A}$ is equal to the multiplicity of the root* 1 *of the characteristic equation of $\bar{A}$.*

In conformity with the notation employed in § 2, let

$$(71)\qquad \bar{\Omega}(\bar{A}) \equiv \bar{A}^{\mu} + \bar{p}_1\bar{A}^{\mu-1} + \ldots + \bar{p}_{\mu-1}\bar{A} = 0$$

be the syzygy of lowest order in powers of $\bar{A}$. Then $\rho\,\phi(\rho)$ contains $\bar{\Omega}(\rho)$. Whence it follows that n is the maximum number of distinct non-zero roots of the equation $\bar{\Omega}(\rho) = 0$. Therefore, by *theorem III*, and what was proved p. 636, *$\bar{A}$ is nilpotent if, for some positive integer p,*

$$\bar{S}\,\bar{A}^{p+h} = 0 \qquad (h = 0, 1, 2, \ldots n - 1).$$

Conversely, by *theorem I, if $\bar{A}$ is nilpotent, these equations are satisfied for any positive integer p.*

For the scalar functions defined in § 1 of any number $A = \sum_{i=1}^{n} a_i e_i$ of the system $(e_1, e_2, \ldots e_m)$ I shall write S_1A and S_2A as in § 1 and § 2. The symbol $\bar{S}$ also is significant when prefixed to any letter denoting a number of the system $(e_1, e_2, \ldots e_m)$, since any such number belongs to the quadrate ϵ_{uv} $(u, v, = 1, 2, \ldots n)$. We have, by (62) and (67),

$$(72)\qquad \bar{S}e_i = \frac{1}{n}\sum_{u=1}^{n} \theta_{uu}{}^{(i)} \qquad (i = 1, 2, \ldots n);$$

and, therefore

$$(73)\qquad \bar{S}A = \sum_{i=1}^{m} a_i\bar{S}e_i = \frac{1}{n}\sum_{i=1}^{m}\sum_{u=1}^{n} a_i\theta_{uu}{}^{(i)}.$$

Let now

$$(74)\qquad X = \sum_{i=1}^{m} x_i e_i = \sum_{i=1}^{m}\sum_{u=1}^{n}\sum_{v=1}^{n} x_i\theta_{uv}{}^{(i)}\epsilon_{uv};$$

and let the number system $(e_1, e_2, \dots e_m)$ contain at least one number satisfying the system of equations

$$(75) \qquad \bar{S}Xe_i = x_1\bar{S}e_1e_i + x_2\bar{S}e_2e_i + \dots + x_m\bar{S}e_me_i = 0 \qquad (i = 1, 2, \dots m).$$

$$(76) \qquad \nabla \equiv \begin{vmatrix} \bar{S}e_1e_1, & \bar{S}e_2e_1, & \dots & \bar{S}e_me_1 \\ \bar{S}e_1e_2, & \bar{S}e_2e_2, & \dots & \bar{S}e_me_2 \\ \cdots & \cdots & \cdots & \cdots \\ \bar{S}e_1e_m, & \bar{S}e_2e_m, & \dots & \bar{S}e_me_m \end{vmatrix},$$

we, therefore, now have $\nabla = 0$, Let $X = B$ be any number of $(e_1, e_2, \dots e_m)$ satisfying equations (75). Then B is nilpotent; moreover, the product, in either order, of B and any number $A = \sum_{k=1}^{m} a_k e_k$ of the system $(e_1, e_2, \dots e_m)$ is also a solution of equations (75). For, for any number

$$Y = y_1e_1 + y_2e_2 + \dots + y_me_m$$

of the system $(e_1, e_2, \dots e_m)$, we now have

$$\bar{S}BY = y_1\bar{S}Be_1 + y_2\bar{S}Be_2 + \dots + y_m\bar{S}Be_m = 0:$$

wherefore, in particular,

$$\bar{S}B^{2+h} = \bar{S}BB^{h+1} = 0 \qquad (h = 1, 2, \dots n - 1),$$

and thus, by the theorem given on p. 654, B is nilpotent; further,

$$\bar{S}(BA\cdot e_i) = \bar{S}(B\cdot Ae_i) = 0,$$
$$\bar{S}(AB\cdot e_i) = \bar{S}(e_i\cdot AB) = \bar{S}(e_iA\cdot B) = \bar{S}(B\cdot e_iA) = 0 \qquad (i = 1, 2, \dots m).$$

Since both BA and AB are solutions of equations (75), it follows by what has just been proved that both BA and AB are nilpotent. In particular, for $1 \leqq i \leqq m$, Be_i is nilpotent; and, therefore, by *theorem I*, $S_1Be_i = 0$. Whence it follows that B is a solution of the system of equations

$$(77) \qquad S_1Xe_i = x_1S_1e_1e_i + x_2S_1e_2e_i + \dots + x_mS_1e_me_i = 0 \qquad (i = 1, 2, \dots m).$$

Wherefore, we now have

$$\Delta_1 = \Delta_2 = 0.^{21}$$

Conversely, if $B = \sum_{k=1}^{m} b_k e_k$ is any solution of equations (77), Be_j $(1 \leqq j \leqq m)$ is by *theorem II* then also a solution of these equations, and thus Be_j, by *theorem I*, is nilpotent: therefore, by the theorem of p. 654, $\overline{S}Be_j = 0$ for $j = 1, 2, \dots m$; that is, B is a solution of equations (75). Let the nullity of ∇ be m', where $1 \leqq m' \leqq m$. There is then a set of just m' linearly independent numbers $B_1, B_2, \dots B_{m'}$ of the system $(e_1, e_2, \dots e_m)$ satisfying equations (75); therefore, just m' linearly independent numbers of this system satisfying equations (77): whence it follows that the nullity of Δ_1 is m'. And since each of the B's satisfies equations (77) it follows, from *theorem II*, that $B_1, B_2, \dots B_{m'}$ constitute an invariant nilpotent sub system of $(e_1, e_2, \dots e_m)$ containing every invariant nilpotent sub system of $(e_1, e_2, \dots e_m)$.

Let now $\nabla \neq 0$. In this case, if, for any two numbers

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of $(e_1, e_2, \dots e_m)$, we have

$$\overline{S}Ae_i = \overline{S}Be_i \qquad (i = 1, 2, \dots m),$$

then $A = B$; otherwise, there is a number $A - B \neq 0$ of the system satisfying equations (77). In this case, $\Delta_1 \neq 0$ and the number system $(e_1, e_2, \dots e_m)$ contains a nodulus but no invariant nilpotent sub system.

Let now the number system $(e_1, e_2, \dots e_m)$ be transformed by the substitution

$$(78) \qquad e'_i = \tau_{i1}e_1 + \tau_{i2}e_2 + \dots + \tau_{im}e_m \qquad (i = 1, 2, \dots m);$$

and let

$$(79) \qquad \nabla' \equiv \left| \begin{array}{l} \overline{S}e'_i e'_j \\ (i, j = 1, 2, \dots m) \end{array} \right| .$$

Then, since

$$\overline{S}e'_i e'_j = \sum_{h=1}^{m} \sum_{k=1}^{m} \tau_{ih}\tau_{jk}\overline{S}e_h e_k \qquad (i, j = 1, 2, \dots m),$$

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[21] See p. 630.

we have

$$\nabla' = T^2 \nabla, \tag{80}$$

where T is the determinant of the transformation. Therefore, the equation $\nabla = 0$ is invariant to any transformation of the units of the system $(e_1 e_2, \dots e_m)$.

We have now the following theorem:

Theorem V. *Let* $(e_1, e_2, \dots e_m)$ *be any given number system constituting a sub system of the quadrate* ϵ_{uv} $(u, v = 1, 2, \dots u)$: *thus let*

$$e_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \theta_{uv} \epsilon_{uv} \qquad (i = 1, 2, \dots m).$$

For any given number

$$\bar{A} = \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv} \epsilon_{uv}$$

of the quadrate, let

$$\bar{S}\bar{A} = \frac{1}{n} \sum_{u=1}^{n} a_{uu},$$

when, for any given number

$$X = \sum_{i=1}^{m} x_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} x_i \theta_{uv}^{(i)} \epsilon_{uv}$$

of the system $(e_1, e_2, \dots e_m)$, *we have*

$$\bar{S}X = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} x_i \theta_{uu}^{(i)}.$$

Let

$$\nabla \equiv \begin{vmatrix} \bar{S} e_i e_j \\ (i, j = 1, 2, \dots m) \end{vmatrix}$$

denote the resultant of the system of equations

$$\bar{S} X e_i = \sum_{j=1}^{m} x_j \bar{S} e_j e_i = 0. \qquad (i = 1, 2, \dots m).$$

Then, if the number system be transformed by the substitution

$$e_i' = \tau_{i1}e_1 + \tau_{i2}e_2 + \ldots + \tau_{im}e_m \qquad (i = 1, 2, \ldots m),$$

and if

$$\nabla' \equiv \left| \begin{matrix} \overline{S}e'_ie'_j \\ (i, j = 1, 2, \ldots m) \end{matrix} \right|,$$

we have

$$\nabla' = T^2\nabla,$$

where T is the determinant of the substitution. If $\nabla \neq 0$, the system $(e_1, e_2, \ldots e_m)$ contains a modulus but no invariant nilpotent sub system; and, in this case, if for any two numbers

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of the system we have

$$\overline{S}Ae_i = \overline{S}Be_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$. If $\nabla = 0$ and m' $(0 < m' \leqq m)$ is the nullity of ∇, the system $(e_1, e_2, \ldots e_m)$ contains a maximum invariant nilpotent sub system with m' units constituted by any m' linearly independent solutions of the equations $\overline{S}Xe_i = 0$ $(i = 1, 2, \ldots m)$.

In precisely the same way we may now prove the following theorem of which the preceding theorem is a special case:

Theorem VI. *Let $(e_1, e_2, \ldots e_m)$ be any given hyper complex number system constituting a sub system of the number system $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ whose constants of multiplication are $\overline{\gamma}_{uvw}$ for $u, v, w = 1, 2, \ldots n$, so that*

$$\epsilon_u\epsilon_v = \sum_{w=1}^{n} \overline{\gamma}_{uvw}\epsilon_w;$$

and let

$$e_i = \sum_{u=1}^{n} \theta_{iu}\epsilon_u \qquad (i = 1, 2, \ldots m).$$

For any number $\overline{A} = \sum_{u=1}^{n} \overline{a}_u\epsilon_u$ of the system $(\epsilon_1, \epsilon_2, \ldots \epsilon_n)$, let

$$\overline{S}_1\overline{A} = \frac{1}{n}\sum_{u=1}^{n}\sum_{v=1}^{n} \overline{a}_u\overline{\gamma}_{uvv}, \qquad \overline{S}_2\overline{A} = \frac{1}{n}\sum_{u=1}^{n}\sum_{v=1}^{n} \overline{a}_u\overline{\gamma}_{vuv},$$

in which case, for any number

$$A = \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{n} a_i \theta_{iu} \epsilon_u$$

of the system $(e_1, e_2, \ldots e_m)$, *we have*

$$\bar{S}_1 A = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} a_i \theta_{iu} \bar{\gamma}_{uvv}$$

$$\bar{S}_2 A = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} a_i \theta_{iu} \bar{\gamma}_{vuv}.$$

Finally, let $X = \sum_{i=1}^{m} x_i e_i$, *and let*

$$\nabla_1 \equiv \left| \begin{array}{l} \bar{S}_1 e_i e_j \\ (i\, j, = 1, 2, \ldots m) \end{array} \right|, \qquad \nabla_2 = \left| \begin{array}{l} \bar{S}_2 e_i e_j \\ (i, j = 1, 2, \ldots m) \end{array} \right|$$

be, respectively, the resultants of the systems of equations

$$(\alpha) \quad \bar{S}_1 X e_i = x_1 \bar{S}_1 e_1 e_i + x_2 \bar{S}_1 e_2 e_i + \ldots + x_m \bar{S}_1 e_m e_i = 0 \quad (i = 1, 2, \ldots m),$$

$$(\beta) \quad \bar{S}_2 X e_i = x_1 \bar{S}_2 e_2 e_i + x_2 \bar{S}_2 e_2 e_i + \ldots + x_m \bar{S}_2 e_m e_i = 0 \quad (i = 1, 2, \ldots m).$$

Then, if the number system $(e_1, e_2, \ldots e_m)$ *is transformed by the substitution*

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \quad (i = 1, 2, \ldots m),$$

and if

$$\nabla'_1 \equiv \left| \begin{array}{l} \bar{S}_1 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{array} \right|, \qquad \nabla'_2 \equiv \left| \begin{array}{l} \bar{S}_2 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{array} \right|$$

we have

$$\nabla_1' = T^2 \nabla_1, \qquad \nabla'_2 = T^2 \nabla_2,$$

where T is the determinant of the substitution. If $\nabla_1 \neq 0$, *in which case* $\nabla_2 \neq 0$, *and conversely, the system* $(e_1, e_2, \ldots e_m)$ *contains a modulus, but no invariant nilpotent sub system; and in this case, if for any two numbers*

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of this system, we have

$$\bar{S}_1 A e_i = \bar{S}_1 B e_i \qquad (i = 1, 2, \ldots m),$$

or

$$\bar{S}_2 A e_i = \bar{S}_2 B e_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$. *If the nullity of* ∇_1 *is* m' $(0 < m' \leqq m)$, *in which case* $\nabla_1 = 0$, *the nullity of* ∇_2 *is* m', *and conversely; and the system* $(e_1, e_2, \ldots e_m)$ *then contains a maximum invariant nilpotent sub system constituted by any* m' *linearly independent numbers of* $(e_1, e_2, \ldots e_m)$ *satisfying equations* (α), *or equations* (β), *every solution of equations* (α) *being a solution of equations* (β), *and conversely.*

§ 4.

Let $(e_1, e_2, \ldots e_m)$ be any given number system; let ϵ_{uv}, for $u, v = 1, 2, \ldots n$, constitute a quadrate of which $(e_1, e_2, \ldots e_m)$ is a sub system; and let

$$(81) \qquad e_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \theta_{uv}^{(i)} \epsilon_{uv} \quad (i = 1, 2, \ldots m).$$

The units of the system $(e_1, e_2, \ldots e_m)$ are then represented, respectively, or may be identified, respectively, with the m linearly independent matrices defined by equations (63).

The number system $(e'_1, e'_2, \ldots e'_m)$ reciprocal to $(e_1, e_2, \ldots e_m)$ is then also a sub system of the quadrate: that is,

$$(82) \qquad e'_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \eta_{uv}^{(i)} \epsilon_{uv} \qquad (i = 1, 2, \ldots m)$$

for a proper choice of the η's. For the m numbers $e'_1, e'_2, \ldots e'_m$ of the quadrate defined by equations (82) may be identified, respectively, with the m matrices $E'_1, E'_2, \ldots E'_m$, where E'_i, for $1 \leqq i \leqq m$, is defined by the equations

$$(83) \quad E'_i(\xi_1, \xi_2, \ldots \xi_n) = \left(\begin{matrix} \eta_{11}^{(i)}, & \eta_{12}^{(i)}, & \ldots & \eta_{1n}^{(i)} \\ \eta_{21}^{(i)}, & \eta_{22}^{(i)}, & \ldots & \eta_{2n}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{n1}^{(i)}, & \eta_{n2}^{(i)}, & \ldots & \eta_{nn}^{(i)} \end{matrix}\right)(\xi_1, \xi_2, \ldots \xi_n); \quad (i = 1, 2, \ldots m);$$

and, therefore, if we put

$$(84)\qquad \eta_{uv}^{(i)} = \theta_{vu}^{(i)} \qquad (i = 1, 2, \dots m;\ u, v = 1, 2, \dots n),$$

we then have

$$(85)\qquad E'_i = tr.\, E_i^{22} \qquad (i = 1, 2, \dots m):$$

whence it follows that $E'_1, E'_2, \dots E'_m$ are linearly independent, and

$$(86)\quad E'_iE'_j = tr. E_i \cdot tr. E_j = tr. (E_jE_i)$$
$$= tr. (\sum_{k=1}^{m} \gamma_{jik}E_k) = \sum_{k=1}^{m} \gamma_{jik}\, tr.\, E_k = \sum_{k=1}^{m} \gamma_{jik}E'_k$$
$$(i, j = 1, 2, \dots m);$$

that is to say, the numbers $e'_1, e'_2, \dots e'_m$ of the quadrate are then linearly independent, and

$$(87)\qquad e'_ie'_j = \sum_{k=1}^{m} \gamma_{jik}e'_k \qquad (i, j = 1, 2, \dots m).$$

We may take $n = m$, and, at the same time, put

$$(88)\qquad \theta_{uv}^{(i)} = \gamma_{ivu},\ \eta_{uv}^{(i)} = \gamma_{iuv} \qquad (i, u, v = 1, 2, \dots m),$$

unless, for $a_1, a_2, \dots a_m$ not all zero, we have, simultaneously,

$$(89)\qquad \sum_{i=1}^{m} a_i\gamma_{ivu} = \sum_{i=1}^{m} a_i\theta_{uv}^{(i)} = 0,$$
$$(u, v = 1, 2, \dots m),$$

in which case, neither the m matrices $E_1, E_2, \dots E_m$ of order m representing, respectively, $e_1, e_2, \dots e_m$ nor the m matrices $E'_1, E'_2, \dots E'_m$ representing, respectively, $e'_1, e'_2, \dots e'_m$, are linearly independent.[23]

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[22] I here follow Cayley in denoting by *tr. M* the transverse (or conjugate) of any given matrix M. Loc. cit., p. 31.

[23] If $n = m$ and $\theta_{uv}^{(i)} = \gamma_{ivu}$ for $i, u, v = 1, 2, \dots m$, the constituent of E_iE_j in the uth row and vth column is

$$\sum_{w=1}^{m} \theta_{uw}^{(i)}\theta_{wv}^{(j)} = \sum_{w=1}^{m} \gamma_{iwu}\gamma_{jvw} = \sum_{w=1}^{m} \gamma_{ijw}\gamma_{wvu} = \sum_{w=1}^{m} \gamma_{ijw}\theta_{uv}^{(w)}$$

by (54); and, therefore,

$$E_iE_j = \sum_{w=1}^{m} \gamma_{ijw}E_w.$$

In this case, there is some number $A = \sum_{i=1}^{m} a_i e_i \neq 0$ of the system $(e_1, e_2, \ldots e_m)$ such that $A X = 0$ for any number $X = \sum_{i=1}^{m} x_i e_i$ of this system; since we should then have

$$AX = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_i x_j \gamma_{ijk} e_k$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{m} (\sum_{i=1}^{m} a_i \gamma_{ijk}) x_j e_k = 0.$$

Conversely, if $A = \sum_{i=1}^{m} a_i e_i \neq 0$ and $A X = 0$ for every number X of $(e_1, e_2, \ldots e_m)$, equations (89) are satisfied for at least one system of values $a_1, a_2, \ldots a_m$ not all zero, and we cannot assign to the θ's, nor to the η's, the values given by equations (88). In this case, we have

$$S_1 A e_i = 0 = S_2 A e_i \qquad (i = 1, 2, \ldots m);$$

and, therefore,

$$\Delta_1 = \Delta_2 = 0.$$

It is to be noted that equations (89) are the conditions, necessary and sufficient, that the reciprocal system shall contain a number $A' = \sum_{i=1}^{m} a_i e'_i \neq 0$ such that $X'A' = 0$ for any number $X' = \sum_{i=1}^{m} x_i e'_i$ of this system.

Further, we may take $n = m$ and put

$$(90) \qquad \theta_{uv}^{(i)} = \gamma_{uiv}, \; \eta_{uv}^{(i)} = \gamma_{viu} \qquad (i, u, v = 1, 2, \ldots m),$$

unless for $b_1, b_2, \ldots b_m$ not all zero, we have

$$(91) \qquad \sum_{i=1}^{m} b_i \gamma_{uiv} = \sum_{i=1}^{m} b_i \theta_{uv}^{(i)} = 0$$

$$(u, v = 1, 2, \ldots m),$$

in which case $E_1, E_2, \ldots E_m$ are not linearly independent, nor are

$E'_1, E'_2, \dots E'_m$ linearly independent.[24] In this case, there is some number $B = \sum\limits_{i=1}^{m} b_i e_i \neq 0$ of $(e_1, e_2, \dots e_m)$ such that $XB = 0$ for every number $X = \sum\limits_{i=1}^{m} x_i e_i$ of this system; and there is also a number $B' = \sum\limits_{i=1}^{m} b_i e'_i \neq 0$ of the reciprocal system such that $B'X' = 0$ for every number X' of the reciprocal system. Conversely, if there is some number $B = \sum\limits_{i=1}^{m} b_i e_i \neq 0$ of $(e_1, e_2, \dots e_m)$ such that $XB = 0$ for every number X of this system (or if, for $B' = \sum\limits_{i=1}^{m} b_i e'_i \neq 0$ and for any number X' of the reciprocal system, we have $B' X' = 0$) equations (91) are satisfied for some system of values $b_1, b_2, \dots b_m$ not all zero, and we cannot assign to the θ's, nor to the η's, the values given by equations (90). When equations (91) are satisfied,

$$S_1 B e_i = 0 = S_2 B e_i \qquad (i = 1, 2, \dots m);$$

and, therefore,

$$\Delta_1 = \Delta_2 = 0.$$

When the system $(e_1, e_2, \dots e_m)$ contains a modulus it is not possible to satisfy equations (89) nor equations (91).

We may distinguish three cases. *First,* the given number system $(e_1, e_2, \dots e_m)$ may contain both a number $A = \sum\limits_{i=1}^{m} a_i e_i \neq 0$ and a number $B = \sum\limits_{i=1}^{m} b_i e_i \neq 0$ such that $AX = 0$, $XB = 0$ for every number $X = \sum\limits_{i=1}^{m} x_i e_i$ of the system, in which case the system does not contain a modulus and $\Delta_1 = \Delta_2 = 0$. In this case it is not possible to assign to the θ's the values given by either equations (88) or (90), nor to assign to the η's the values given by either of these equations. Nevertheless, it may be possible in this case to put $n = m$, provided $m > 2$, but not otherwise. Thus let $m = 3$, and let

$$e_1^2 = e_1, \qquad e_1 e_2 = 0, \qquad e_1 e_3 = e_3,$$

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[24] If $n = m$ and $\theta_{uv}^{(i)} = \gamma_{uiv}$ for $i, u, v = 1, 2, \dots m$, it follows from (54) that $E_i E_j = \sum\limits_{w=1}^{m} \gamma_{ijw} E_w$. Cf. note 23.

$$e_2e_1 = e_2e_2 = e_2e_3 = 0, \qquad e_3e_1 = e_3e_2 = e_3e_3 = 0:$$

if

$$A = a_2e_2 + a_3e_3 \neq 0, \qquad B = e_2,$$

we have

$$Ae_i = 0, \qquad e_iB = 0 \qquad (i = 1, 2, 3);$$

and we may now put $n = m = 3$, and

$$e_1 = \epsilon_{11}, \qquad e_2 = \epsilon_{23}, \qquad e_3 = \epsilon_{13}.$$

On the other hand, let $m = 2$ and let (e_1, e_2) contain a number $A \neq 0$ such that

$$A\,e_1 = A\,e_2 = 0.$$

In this case, we may, without loss of generality, put $A = e_1$, when we have

$$e_1^2 = 0, \qquad e_1e_2 = 0.$$

If now

$$e_i = \begin{pmatrix} \theta_{11}^{(i)}, & \theta_{12}^{(i)} \\ \theta_{21}^{(i)}, & \theta_{22}^{(i)} \end{pmatrix} \qquad (i = 1, 2),$$

we then have, since $e_1^2 = 0$,

$$\begin{pmatrix} \theta_{11}^{(1)}, & \theta_{12}^{(1)} \\ \theta_{21}^{(1)}, & \theta_{22}^{(1)} \end{pmatrix} = \varpi \begin{pmatrix} 0, & k \\ 0, & 0 \end{pmatrix} \varpi^{-1},$$

where $k \neq 0$ and the determinant of the matrix ϖ is not zero; and, therefore, since $e_1e_2 = 0$,

$$\begin{pmatrix} \theta_{11}^{(2)}, & \theta_{12}^{(2)} \\ \theta_{21}^{(2)}, & \theta_{22}^{(2)} \end{pmatrix} = \varpi \begin{pmatrix} \alpha, & \beta \\ 0, & 0 \end{pmatrix} \varpi^{-1},$$

where, without loss of generality, we may put $\alpha = 1$, $\beta = 0$, giving

$$e_2e_1 = e_1, \qquad e_2^2 = e_2.$$

This system, however, contains no number $B \neq 0$ for which

$$e_1\,B = e_2\,B = 0.$$

Second, the number system $(e_1, e_2, \dots e_m)$ may contain either a number $A \neq 0$ such that $Ae_i = 0$ for $i = 1, 2, \dots m$, or a number $B \neq 0$ such that $e_iB = 0$ for $i = 1, 2, \dots m$, but not both. In this case, we may put $n = m$ and assign to the θ's and η's either the values given by equations (90) or equations (88) respectively.

Third, the system $(e_1, e_2, \dots e_m)$ may contain neither a number $A \neq 0$ such that $Ae_i = 0$ for $i = 1, 2, \dots m$ nor a number $B \neq 0$ such that $e_iB = 0$ for $i = 1, 2, \dots m$, for which a sufficient, but not necessary condition, is the existence of a modulus, and, *a fortiori*, that $\Delta_1 \neq 0$. In this case, we may put $n = m$ and assign to the θ's the values given by equations (88), and to the η's the values given by equations (90). We then have

$$\begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{ivu} \epsilon_{uv}, \\ A' &= \sum_{i=1}^{m} a_i e'_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{viu} \epsilon_{uv}; \end{aligned} \tag{92}$$

and, therefore,

$$\begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{iuu} = \overline{S} A, \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{uiu} = \overline{S} A'. \end{aligned} \tag{93}$$

On the other hand, if we assign to the θ's the values given by (90) and to the η's the values given by (88), which is now possible, we shall have

$$\begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{uiv} \epsilon_{uv}, \\ A' &= \sum_{i=1}^{m} a_i e'_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{iuv} \epsilon_{uv}; \end{aligned} \tag{94}$$

whence follows

$$\begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{iuu} = \overline{S} A', \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{uiu} = \overline{S} A. \end{aligned} \tag{95}$$

When either the representation of the number system $(e_1\, e_2, \ldots e_m)$ and its reciprocal system given by equations (88) or by equations (90) fails, and indeed in any case, we may proceed as follows. Let $n = m + 1$, and let

$$\theta_{uv}^{(i)} = \gamma_{ivu}, \qquad \theta_{m+1,v}^{(i)} = \theta_{m+1,m+1}^{(i)} = 0 \qquad (i,\, u,\, v = 1, 2, \ldots m), \tag{96 a}$$

$$\theta_{u,m+1}^{(i)} = 0, \qquad \theta_{i,m+1}^{(i)} = 1 \qquad (i,\, u = 1, 2, \ldots m;\ u \neq i); \tag{96 b}$$

moreover, let

$$\eta_{uv}^{(i)} = \gamma_{viu}, \qquad \eta_{m+1,v}^{(i)} = \eta_{m+1,m+1}^{(i)} = 0 \qquad (i,\, u,\, v = 1, 2, \ldots m), \tag{97 a}$$

$$\eta_{u,m+1}^{(i)} = 0 \qquad \eta_{i,m+1}^{(i)} = 1 \qquad (i,\, u = 1, 2, \ldots m;\ u \neq i). \tag{97 b}$$

The m matrices $E_1, E_2, \ldots E_m$ which we thus obtain have the same multiplication table as the units of the system $(e_1, e_2, \ldots e_m)$ and are, moreover, linearly independent. For, if

$$c_1 E_1 + c_2 E_2 + \ldots + c_m E_m = 0,$$

then

$$\sum_{i=1}^{m} c_i \theta_{uv}^{(i)} = 0 \qquad (u,\, v = 1, 2, \ldots m + 1);$$

and, therefore, in particular

$$c_u = \sum_{i=1}^{m} c_i \theta_{u,m+1}^{(i)} = 0 \qquad (u = 1, 2, \ldots m).$$

Further, the m matrices determined by the above values of the η's are also linearly independent and have the same multiplication table as the system $(e'_1, e'_2, \ldots e'_m)$ reciprocal to $(e_1, e_2, \ldots e_m)$. We now have

$$\begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} a_i \Big(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{ivu} e_{uv} + e_{i,m+1} \Big), \\ A' &= \sum_{i=1}^{m} a_i e_i' = \sum_{i=1}^{m} a_i \Big(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{viu} \epsilon_{uv} + \epsilon_{i,m+1} \Big); \end{aligned} \tag{98}$$

and, therefore,

$$\begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{iuu} = \frac{m+1}{m} \bar{S} A, \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{uiu} = \frac{m+1}{m} \bar{S} A'. \end{aligned} \tag{99}$$

We may also proceed as follows. Let $n = m + 1$, and let

$$\theta_{uv}^{(i)} = \gamma_{uiv}, \qquad \theta_{u,m+1}^{(i)} = \theta_{m+1,m+1}^{(i)} = 0 \qquad (i,\, u,\, v = 1, 2, \ldots m), \tag{100 a}$$

$$\theta_{m+1,v}{}^{(i)} = 0, \qquad \theta_{m+1,i}{}^{(i)} = 1 \qquad (i,\, v = 1, 2, \ldots m;\; v \neq i); \tag{100 b}$$

moreover, let

$$\eta_{uv}{}^{(i)} = \gamma_{iuv}, \qquad \eta_{u,m+1}{}^{(i)} = \eta_{m+1,m+1}{}^{(i)} = 0 \qquad (i,\, u,\, v = 1, 2, \ldots m), \tag{101 a}$$

$$\eta_{m+1,v}{}^{(i)} = 0, \qquad \eta_{m+1,i}{}^{(i)} = 1 \qquad (i,\, v = 1, 2, \ldots m;\; v \neq i). \tag{101 b}$$

The m matrices $E_1, E_2, \ldots E_m$ thus obtained are linearly independent, as are also the m matrices $E'_1, E'_2, \ldots E'_m$; and the former have the same multiplication table as the units of the system $(e_1, e_2, \ldots e_m)$, while the latter have the same multiplication table as the units of the reciprocal system. We now have

$$\begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} a_i \left(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{uiv} \epsilon_{uv} + \epsilon_{m+1,i} \right) \\ A' &= \sum_{i=1}^{m} a_i e_i' = \sum_{i=1}^{m} a_i \left(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{iuv} \epsilon_{uv} + \epsilon_{m+1,i} \right); \end{aligned} \tag{102}$$

and, therefore,

$$\begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} \gamma_{iuu} = \frac{m+1}{m} \bar{S} A', \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} \gamma_{uiu} = \frac{m+1}{m} \bar{S} A. \end{aligned} \tag{103}$$

The fundamental properties of the scalar functions given in *theorem I* are more readily proved for the special case in which the number system is a quadrate than in the general case. What precedes in this section indicates how the properties of these functions may be made to depend upon the properties of the single scalar function of a quadrate.

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